

§5. RIEMANN-ROCH THEOREM (FINAL FORM). SERRE DUALITY

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **27 (1981)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **21.07.2024**

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systems at P and Q , f is the map $Z \rightarrow Z^{e_p+1} = w$ of the unit disc $U \subset \mathbf{C}$ onto another copy W of it. Since $1, Z, \dots, Z^{e_p}$ provide an \mathcal{O}_W -basis for $f_0(Q_U)$, the value of δ on a local generator of $\mathcal{L} \otimes \mathcal{L}$ is given by

$$\det(\tau(Z^{i+j})), \quad 0 \leq i, j \leq e = e_p.$$

But

$$\tau(Z^{i+j}) = Z^{i+j} (1 + \zeta^{i+j} + (\zeta^{i+j})^2 + \dots + (\zeta^{i+j})^e),$$

(ζ denoting a primitive $(e+1)$ -st root of unity), hence

$$\begin{aligned} \tau(Z^{i+j}) &= (e+1)Z^{i+j} \quad \text{if } i+j = 0 \quad \text{or } e+1, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Hence $\det(\tau(Z^{i+j}))$ is a (nonzero) constant multiple of $Z^{(e+1)e} = w^e$ as asserted.

If $f^{-1}(Q)$ consists of several points, the situation is a direct sum of those considered above, and δ is indeed as asserted. This proves Theorem (4.1).

(4.5) *Remark.* Let the notation be as above, and let $E(X)$ denote the topological Euler-Poincaré characteristic of X . Then, using the formula $E(X) = \text{number of vertices} - \text{number of edges} + \text{number of faces}$ in any triangulation of X , it is easy to see that $E(X) = rE(Y) - \deg R(Y=\mathbf{P}^1)$. Indeed, choose any triangulation of Y which contains all the images of the ramification points of f as vertices, and lift it to a triangulation of X . Then, while r edges or faces lie over each edge or face of Y , the ramification points reduce the number of vertices over certain vertices of Y , and one gets the formula asserted. Since $E(Y) = 2$, (4.2) yields:

(4.6) **COROLLARY.** $\deg K_X = -E(X) = 2g - 2$, i.e. g is also the topological genus $(1/2)b_1(X)$ of the compact oriented surface X .

§ 5. RIEMANN-ROCH THEOREM (FINAL FORM). SERRE DUALITY

(5.1) **(RIEMANN-ROCH THEOREM).** For any line bundle \mathcal{L} on X ,

$$h^0(\mathcal{L}) - h^0(K \otimes \mathcal{L}^{-1}) = \deg \mathcal{L} - g + 1.$$

Proof: It is enough to prove

(5.2) for all \mathcal{L} , $h^0(\mathcal{L}) - h^0(K \otimes \mathcal{L}^{-1}) \geq \deg \mathcal{L} - g + 1$. For then, replacing \mathcal{L} by $K \otimes \mathcal{L}^{-1}$ changes only the sign of the left side, and the same is true of the right side by (4.1) (cf. [4], p. 147).

Now (5.2) is true if $\deg \mathcal{L} > \deg K$, for then $h^0(K \otimes \mathcal{L}^{-1}) = 0$, and we can use (3.5). Thus, to prove (5.2), we may assume that $\mathcal{L} = \mathcal{O}(D)$ for some $D \in \text{Div } X$, and that (5.2) holds for $\mathcal{L}' = \mathcal{O}(D + P_0)$, $P_0 \in X$. Now it is clear that $h^0(\mathcal{L}') \leq h^0(\mathcal{L}) + 1$, and similarly $h^0(K \otimes \mathcal{L}'^{-1}) \leq h^0(K \otimes \mathcal{L}^{-1}) + 1$ (cf. the proof of (3.4)). So (5.2) fails for \mathcal{L} if and only if (*) $h^0(\mathcal{L}') = h^0(\mathcal{L}) + 1$, and $h^0(K \otimes \mathcal{L}^{-1}) = h^0(K \otimes \mathcal{L}'^{-1}) + 1$. But if (*) holds, there exist

$$\sigma \in H^0(X, \mathcal{L}') - H^0(X, \mathcal{L})$$

and

$$\omega \in H^0(X, K \otimes \mathcal{L}^{-1}) - H^0(X, K \otimes \mathcal{L}'^{-1}),$$

and then

$$\sigma\omega = \sigma \otimes \omega \in H^0(X, K \otimes \mathcal{O}(P_0)) - H^0(X, K),$$

i.e. $\sigma\omega$ is a meromorphic form with precisely one simple pole at P_0 . But this is impossible: if D is a disc around P_0 in some coordinate system centred at P_0 , then $\int_{\partial D} \sigma\omega = - \int_{\partial(X-D)} \sigma\omega = 0$ by Stokes' theorem, while $\int_{\partial D} \sigma\omega \neq 0$ by the Residue theorem. Thus (*) cannot hold, and (5.2) is proved, q.e.d.

(5.3) COROLLARY. For any line bundle \mathcal{L} on X , $h^1(\mathcal{L}) = h^0(K \otimes \mathcal{L}^{-1})$.

Proof: Compare (5.1) and (3.4).

(5.4) COROLLARY. $h^0(K) = g$ and $h^1(K) = 1$.

Before proceeding to Serre duality, we examine the notion of residue in greater detail. Thus let $U \subset X$ be open, and ω a meromorphic 1-form on U with a pole at $P \in U$. Then, in terms of a uniformising parameter t at P , $\omega = f dt$ near P , with f a meromorphic function at P . The residue of ω at P is $\frac{1}{2\pi i}$ times the coefficient of $1/t$ in the Laurent expansion of f in powers of t . The independence of $\text{Res}_P(\omega)$ on the choice of t can be proved either by direct computation or by identifying it with $1/2\pi i \int_{\gamma} \omega$, where γ is a suitable curve around P . By the argument already used above (Stokes' theorem), one gets

(5.5) (RESIDUE THEOREM). *The sum of the residues of any meromorphic 1-form on X is zero.*

(5.6) COROLLARY. *Given distinct $P, Q \in X$, there exists a meromorphic 1-form on X , holomorphic outside P and Q , and with simple poles at P, Q of residue 1 and -1 respectively.*

Proof: Let $\mathcal{L} = K \otimes \mathcal{O}(P+Q)$. Then $\deg K \otimes \mathcal{L}^{-1} < 0$, hence $h^0(\mathcal{L}) = g + 1$ by (5.1), i.e. there exists $\omega \in H^0(X, \mathcal{L}) - H^0(X, K)$. Then it is clear that the residues of ω at P and Q must be non-zero, while their sum is zero (by (5.5)), hence a suitable constant multiple of ω will have the desired properties.

(5.7) PROPOSITION. *There is a canonical isomorphism $\text{res} : H^1(X, K) \rightarrow \mathbf{C}$.*

Proof: Pick any $P \in X$, and a coordinate neighbourhood U of P . Let \mathcal{U} be the covering $\{U, X - P\}$ of X . Then, by taking residues at P , we get a map $\text{res}_P : Z^1(\mathcal{U}, K) \rightarrow \mathbf{C}$. This map is not zero, and induces a map $H^1(\mathcal{U}, K) \rightarrow \mathbf{C}$ (by the residue theorem). Since $h^1(K) = 1$, $\text{res}_P : H^1(\mathcal{U}, K) \rightarrow H^1(X, K) \rightarrow \mathbf{C}$ is in fact an isomorphism. That the map $\text{res}_P : H^1(X, K) \rightarrow \mathbf{C}$ is independent of the choice of $P \in X$ is precisely the meaning of (5.6), and we get the asserted canonical isomorphism res .

(5.8) SERRE DUALITY. *For any line bundle \mathcal{L} on X , the natural bilinear form*

$$\zeta : H^0(X, \mathcal{L}) \times H^1(X, K \otimes \mathcal{L}^{-1}) \rightarrow H^1(X, K) \xrightarrow{\text{res}} \mathbf{C}$$

is nondegenerate.

(5.9) Remark. For any covering \mathcal{U} of X , the natural map $\mathcal{L} \times (K \otimes \mathcal{L}^{-1}) \rightarrow K$ defines an obvious pairing

$$H^0(X, \mathcal{L}) \times Z^1(\mathcal{U}, K \otimes \mathcal{L}^{-1}) \rightarrow Z^1(\mathcal{U}, K)$$

which is easily seen to induce the pairing

$$H^0(X, \mathcal{L}) \times H^1(X, K \otimes \mathcal{L}^{-1}) \rightarrow H^1(X, K)$$

figuring in (5.8).

Proof of (5.8). Since we already know that

$$h^0(X, \mathcal{L}) = h^1(X, K \otimes \mathcal{L}^{-1}),$$

we need only show that, if $\sigma \in H^0(X, \mathcal{L})$ is such that $\zeta(\sigma \otimes \gamma) = 0$ for all $\gamma \in H^1(X, K \otimes \mathcal{L}^{-1})$, then $\sigma \equiv 0$. Now choose any $P \in X$, and a coordinate neighbourhood (U, z) of P centred at P such that $\mathcal{L}|_U \approx \mathcal{O}_U$. Then the covering $\mathfrak{U} = \{U, X - P\}$ is a Leray covering for \mathcal{L}, K and $K \otimes \mathcal{L}^{-1}$ ((3.7)). The $z^n dz, n \in \mathbf{Z}$, can all be regarded as elements of $Z^1(\mathfrak{U}, K \otimes \mathcal{L}^{-1})$; let γ_n denote their images in $H^1(X, K \otimes \mathcal{L}^{-1})$. Then clearly $\rho(\sigma \otimes \gamma_n) = 0$ for all n implies that all the coefficients of the Taylor expansion of σ at P with respect to vanish, hence $\sigma \equiv 0$, q.e.d.

(5.9) SERRE DUALITY FOR VECTOR BUNDLES. *For any vector bundle \mathcal{V} on X , let $\mathcal{V}^* = \text{Hom } \mathcal{O}_X(\mathcal{V}, \mathcal{O}_X)$. Then the natural pairing*

$$\zeta : H^0(X, \mathcal{V}) \times H^1(X, K \otimes \mathcal{V}^*) \rightarrow H^1(X, K) \xrightarrow{\text{res}} \mathbf{C}$$

is non-degenerate.

Proof: Arguing as in the proof of (5.8) we see that the map $H^0(X, \mathcal{V}) \rightarrow (H^1(X, K \otimes \mathcal{V}^*))^*$ induced by ζ is injective, hence $h^0(X, \mathcal{V}) \leq h^1(X, K \otimes \mathcal{V}^*)$. Replacing \mathcal{V} by $K \otimes \mathcal{V}^*$, we also get $h^0(K \otimes \mathcal{V}^*) \leq h^1(\mathcal{V})$. But, by induction on rank \mathcal{V} , we easily deduce from (5.3) that $\chi(K \otimes \mathcal{V}^*) = -\chi(\mathcal{V})$, hence $h^0(X, \mathcal{V}) = h^1(X, K \otimes \mathcal{V}^*)$. Thus ζ is non-degenerate as before.

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(Reçu le 10 juillet 1980)

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