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Autor: Evans, Ronald J.

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3. Theorems of Stickelberger and Davenport-Hasse

We will make use of the following three classical formulas. First [3, (0.8)],

$$G_{f m}\left(\chi^{\frac{q^m-1}{q-1}}\right) = G_f(\chi)^m, \tag{6}$$

where χ is a character on $GF(q^m)$. Next [3, (0.9)],

$$1 = \frac{\chi^l(l)}{G_f(\chi^l)} \prod_{j=0}^{l-1} \frac{G_f(\chi\psi^j)}{G_f(\psi^j)}, \tag{7}$$

where χ, ψ are characters on $GF(q)$ and ψ has order l. Finally [8], [5, p. 25]

$$\frac{G_f(\chi^\alpha)}{(\zeta-1)^{s(\alpha)}} \equiv \frac{1}{\gamma(\alpha)} \equiv \frac{(\zeta-1)^{\alpha-s(\alpha)}}{\alpha!} \pmod{P}, \tag{8}$$

where α is an integer, $0 \leqslant \alpha < q-1$; $s(\alpha)$ denotes the sum of the p-adic digits of α; $\gamma(\alpha)$ denotes the product of the factorials of the p-adic digits of α; P is a prime ideal above p in the ring $\mathcal{O} = Z[\omega]$, where $\omega = \exp(2\pi i/p(q-1))$; and χ is the character on $\mathcal{O}/P \approx GF(q)$ of order $q-1$ which maps the coset $\omega + P$ to $\bar{\omega}$.

4. Proof of (2)

Let η denote the right side of (2). We must show that $\eta = 1$. Let $\delta = \dfrac{q^n-1}{q-1}$, $\theta = w^{k-1}(cw+i_j)$. Using (6), we have

$$\eta^n = \frac{\chi^{ln}(l)\, G_{fn}(\chi^\delta)}{G_{fn}(\chi^{\delta l})} \prod_{j=1}^{e} \prod_{k=1}^{r} \prod_{c=1}^{w^{r-k}} \frac{G_{fn}^n(\chi^\delta\psi^\theta)}{G_{fn}^n(\psi^\theta)}.$$

Consider a fixed pair j, k. For each $a \in \{1, 2, \ldots, n\}$, $G_{fn}(\psi^\theta) = G_{fn}(\psi^{\theta q^a})$, so

$$\prod_{c=1}^{w^{r-k}} G_{fn}(\psi^\theta) = \prod_{c=1}^{w^{r-k}} G_{fn}\left(\psi^{w^{k-1}(cw+ijq^a)}\right).$$

Similarly,

$$\prod_{c=1}^{w^{r-k}} G_{fn}(\chi^\delta\psi^\theta) = \prod_{c=1}^{w^{r-k}} G_{fn}\left(\chi^\delta\psi^{w^{k-1}(cw+ijq^a)}\right).$$