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Thus

$$(9) \quad \eta^n = \frac{\chi^{ln}(l) G_{fn}(\chi^\delta)}{G_{fn}(\chi^{\delta l})} \prod_{j=1}^{l-1} \frac{G_{fn}(\chi^\delta \psi^j)}{G_{fn}(\psi^j)}.$$

Since $n \equiv \delta \pmod{q-1}$, $\chi^{ln}(l) = \chi^{\delta l}(l)$. Therefore, by (7), the right side of (9) equals 1, so

$$(10) \quad \eta^n = 1.$$

By the definition of η and of Gauss sums,

$$\eta^l \equiv \frac{\chi^{l^2}(l) \bar{\chi}^l(l) G_f(\chi^l)}{G_f^l(\chi^l)} \prod_{j=1}^e \prod_{k=1}^r \prod_{c=1}^{w^{r-k}} \frac{\bar{\chi}^{\delta l}(l) G_{fn}(\chi^{\delta l})}{1} \pmod{w},$$

so

$$\eta^l \equiv \frac{\chi^{l^2-l-l\delta(l-1)/n}(l) G_{fn}^{(l-1)/n}(\chi^{\delta l})}{G_f^{l-1}(\chi^l)} \pmod{w}.$$

By (6), $G_{fn}(\chi^{\delta l}) = G_f^n(\chi^l)$; hence

$$(11) \quad \eta^l \equiv 1 \pmod{w}.$$

Thus w divides the norm $N(\eta^l - 1)$. By (10), η^l is an n -th root of unity. Thus if $\eta^l - 1 \neq 0$, then $N(\eta^l - 1)$ divides n , which contradicts the fact that $w + n$. Therefore $\eta^l = 1 = \eta^n$, so since $(l, n) = 1$, $\eta = 1$.

5. PROOF OF (3)

Let η denote the right side of (3). We assume that $0 < \alpha < q - 1$. To see that this presents no loss of generality, we now show that η is unchanged when α is replaced by $\alpha + (q-1)j$, where j is an integer. Clearly $G_f(\chi^\alpha)$ and $\chi^\alpha(l)$ are unchanged, since the restriction $\chi|_{GF(q)}$ has order $q - 1$. Finally, $G_{fl}(\chi^{\alpha\beta})$ is also unchanged, as

$$(12) \quad G_{fl}(\chi^{\alpha\beta}) = G_{fl}(\chi^{\alpha\beta q^j}) = G_{fl}(\chi^{\beta(\alpha+j(q-1))}),$$

where α_j is defined by $\alpha_j \alpha \equiv j \pmod{l}$, $\alpha_j \geq 0$.

Let $\psi = \chi^{\beta(q-1)}$. Using (6), we have

$$\eta^l = \frac{G_{fl}(\chi^{\alpha\beta l})}{\chi^{\alpha l}(l) G_{fl}^l(\chi^{\alpha\beta})} \prod_{j=1}^{l-1} G_{fl}(\psi^j).$$

For each $j \in \{0, 1, \dots, l - 1\}$, we have, by (12),

$$G_{fl}(\chi^{\alpha\beta}) = G_{fl}(\chi^{\alpha\beta}\psi^j).$$

Thus,

$$(13) \quad \eta^l = \frac{G_{fl}(\chi^{\alpha\beta l})}{\chi^{\alpha l}(l)} \prod_{j=0}^{l-1} \frac{G_{fl}(\psi^j)}{G_{fl}(\chi^{\alpha\beta}\psi^j)}.$$

Since $\chi^{\alpha l}(l) = \chi^{\alpha\beta l}(l)$, the right side of (13) equals 1 by (7), so

$$(14) \quad \eta^l = 1.$$

Let P be the prime ideal above p in $\mathcal{O} = Z[\omega]$, where

$$\omega = \exp(2\pi i/p(q^l - 1)),$$

with P chosen such that χ is the character of order $q^l - 1$ on $\mathcal{O}/P \approx GF(q^l)$ which maps the coset $\omega + P$ to $\bar{\omega}$. To show that $\eta = 1$, it suffices to show that $\eta \equiv 1 \pmod{P}$. For, if $\eta \not\equiv 1 \pmod{P}$, then by (14), the norm $N(\eta - 1)$ divides l ; but if also $\eta \equiv 1 \pmod{P}$, then $p \mid N(\eta - 1)$, which yields the contradiction $p \mid l$.

For any integer x , let $L(x)$ denote the least nonnegative residue of $x \pmod{l}$. For integers $i \geq 0$, define

$$\varepsilon_i = \begin{cases} 1, & \text{if } 1 \leq L(i\alpha) \leq L(\alpha) \\ 0, & \text{otherwise,} \end{cases}$$

and

$$c_i = \varepsilon_i + l^{-1}(\alpha - L(\alpha) + (q - 1)L(-i\alpha)).$$

Note that each c_i is an integer with $0 \leq c_i \leq q - 1$. We have

$$\begin{aligned} l\alpha\beta - l \sum_{i=1}^l c_i q^{i-1} &= \sum_{i=1}^l q^{i-1} (\alpha - lc_i) \\ &= \sum_{i=1}^l q^{i-1} \{-l\varepsilon_i + L(\alpha) - L((1-i)\alpha) + L(-i\alpha)\}. \end{aligned}$$

The expressions in braces are easily seen to vanish. Thus we have the following explicit expansion of $\alpha\beta$ in base q :

$$(15) \quad \alpha\beta = \sum_{i=1}^l c_i q^{i-1}.$$

By (8), (14), and the definition of η ,

$$(16) \quad \eta \equiv (u\gamma(\alpha))^{-1} l^{\alpha\gamma}(\alpha\beta) \pmod{P},$$

where

$$u = \prod_{j=1}^{l-1} \gamma(j(q-1)/l).$$

By (15) and (16),

$$\eta \equiv (u\gamma(\alpha))^{-1} l^\alpha \prod_{i=1}^l \gamma(c_i) \pmod{P}.$$

Thus by the second congruence in (8), there is an integer M such that

$$(17) \quad u\eta \equiv \frac{1}{\alpha!} l^\alpha (\zeta-1)^M \prod_{i=1}^l c_i! \pmod{P}.$$

First suppose that $0 < \alpha < l$. Then by (17) and the definition of c_i ,

$$\begin{aligned} u\eta &\equiv \frac{1}{\alpha!} l^\alpha (\zeta-1)^M \prod_{i=1}^l \left(\frac{q-1}{l} L(-i\alpha) \right)! \prod_{j=1}^\alpha \left(1 + \frac{q-1}{l} (l-j) \right) \\ &\equiv (\zeta-1)^M \prod_{i=1}^l \left(\frac{q-1}{l} L(-i\alpha) \right)! \pmod{P}. \end{aligned}$$

By (14), η is a unit, so again applying the second congruence in (8), we find that

$$u\eta \equiv \prod_{i=1}^l \gamma\left(\frac{q-1}{l} L(-i\alpha)\right) \pmod{P}.$$

Since α is prime to l , the numbers $L(-i\alpha)$ run through a complete residue system \pmod{l} as i runs from 1 to l . Thus, by the definition of u following (16), we obtain the desired result $\eta \equiv 1 \pmod{P}$ in the case $0 < \alpha < l$.

Finally, suppose that $l < \alpha < q-1$. We suppose as induction hypothesis that $\eta' \equiv 1 \pmod{P}$, where η' is obtained from η by replacing α by $\alpha-l$. Then by (17) and the definition of c_i , there is an integer N such that

$$\begin{aligned} \eta &\equiv \eta/\eta' \equiv \frac{1}{\alpha!} (\zeta-1)^N (\alpha-l)! l^l \prod_{i=1}^l c_i \\ &= \frac{1}{\alpha!} (\zeta-1)^N (\alpha-l)! \prod_{i=1}^l \{l\varepsilon_i + \alpha - L(\alpha) + (q-1)L(-i\alpha)\} \pmod{P}. \end{aligned}$$

Since the numbers $\{l\varepsilon_i - L(-i\alpha) + \alpha - L(\alpha)\}$ run through the l numbers $\alpha, \dots, \alpha-l+1$ as i runs from 1 to l , we see that $N = 0$ and $\eta \equiv 1 \pmod{P}$.