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6. Proof of (4)

For characters $\psi_1, \ldots, \psi_m$ on $GF(q)$, define the Jacobi sum

$$J(\psi_1, \ldots, \psi_m) = (-1)^{m+1} \sum_{\substack{x_1, \ldots, x_m \in GF(q) \\ x_1 + \ldots + x_m = 1}} \psi_1(x_1) \ldots \psi_m(x_m).$$

We will use the well-known fact that if $\psi_1 \psi_2 \ldots \psi_m$ is nontrivial, then

$$(18) \qquad J(\psi_1, \ldots, \psi_m) = G(\psi_1 \ldots \psi_m)^{-1} \prod_{i=1}^{m} G(\psi_i).$$

Let S denote the left side of (4). If χ_1, χ_2, or χ_3^2 is trivial, then it is easy to verify (4) directly, with use of (18) and (26) below. Thus assume that χ_1, χ_2, and χ_3^2 are nontrivial. With the change of variables

$$u = xy, \qquad v = x + y,$$

we have

$$S = \sum_{u, v \in GF(q)} \chi_1(u) \chi_2(1+u-v) \chi_3(v^2-4u) \{1 + \phi(v^2-4u)\}.$$

It therefore remains to show that

$$(19) \qquad S_1 = \sum_{u, v} \chi_1(u) \chi_2(1+u-v) \chi_3(v^2-4u) = R(\chi_1, \chi_2, \chi_3\phi).$$

Replace v by $u + 1 - v$ to get

$$(20) \qquad S_1 = \sum_{u, v} \chi_1(u) \chi_2(v) \chi_3(1+u^2+v^2-2u-2v-2uv).$$

Replace u by u/t, and v by v/t, to get

$$(21) \qquad S_2 = -S_1 G(\chi_1\chi_2\chi_3^2)$$
$$= \sum_{t \neq 0} \sum_{u, v} \chi_1(u) \chi_2(v) \chi_3(t^2+u^2+v^2-2ut-2vt-2uv) \zeta^{T(t)}.$$

Since $\chi_1\chi_2\chi_3^2$ is nontrivial, the restriction $t \neq 0$ may be dropped. Then replace t by $t + u + v$ to get

$$S_2 = \sum_{t, u, v} \chi_1(u) \chi_2(v) \chi_3(t^2-4uv) \zeta^{T(t+u+v)}.$$

Replace u by ua and v by vb to get

$$(22) \qquad S_3 = S_2 \overline{G(\chi_1)}\, \overline{G(\chi_2)}$$
$$= \sum_{t} \sum_{a, b, u, v \neq 0} \chi_1(u) \chi_2(v) \chi_3(t^2-4uvab) \zeta^{T(t+a(u-1)+b(v-1))}.$$

Replace a by $a/(4uvb)$ to get

$$S_3 = \sum_t \sum_{a,b,u,v\neq 0} \chi_1(u)\chi_2(v)\chi_3(t^2-a)\zeta^{T\left(t+b(v-1)+\frac{a(u-1)}{4uvb}\right)}.$$

Since χ_1 is nontrivial, the restriction $a \neq 0$ may be dropped. Then replace a by $t^2 - a$ to get

$$S_3 = \sum_{a,t} \sum_{b,u,v\neq 0} \chi_1(u)\chi_2(v)\chi_3(a)\zeta^{T\left(t+b(v-1)+\frac{(1-u)(a-t^2)}{4uvb}\right)}$$

$$= -G(\chi_3)\sum_{u\neq 0,1}\sum_{b,v\neq 0} \chi_1(u)\chi_2(v)\chi_3\left(\frac{4uvb}{1-u}\right)\zeta^{T(b(v-1))}\sum_t \zeta^{T\left(t+\frac{t^2(u-1)}{4uvb}\right)}.$$

The inner sum on t equals $-\zeta^{T\left(\frac{uvb}{1-u}\right)}\phi\left(\frac{4uvb}{u-1}\right)G(\phi)$.

Hence

(23) $$S_4 = S_3\left(G(\chi_3)G(\phi)\chi_3(-1)\right)^{-1}$$

$$= \sum_{u\neq 0,1}\sum_{b,v\neq 0}\chi_1(u)\chi_2(v)\chi_3\phi\left(\frac{4uvb}{u-1}\right)\zeta^{T\left(b(v-1)+\frac{uvb}{1-u}\right)}.$$

Therefore,

$$S_4 = \sum_{u\neq 0,1}\sum_{v\neq 0}\chi_1\chi_3\phi(u)\,\chi_2\chi_3\phi(v)\,\overline{\chi_3\phi}\left(\frac{u-1}{4}\right)\sum_b \chi_3\phi(b)\zeta^{T\left(b(v-1)+\frac{buv}{1-u}\right)}.$$

Since χ_2 and $\chi_3\phi$ are nontrivial,

$$S_4 = -G(\chi_3\phi)\sum_{u,v}\chi_1\chi_3\phi(u)\,\chi_2\chi_3\phi(v)\,\overline{\chi_3\phi}\left(\frac{1-u-v}{4}\right),$$

so

(24) $$S_4 = -\chi_3(4)\,G(\chi_3\phi)\,J(\chi_1\chi_3\phi, \chi_2\chi_3\phi, \overline{\chi_3\phi}).$$

Combining (21)-(24), we get

(25) $$S_1 = \frac{\chi_3(-4)\,G(\chi_3)\,G(\phi)\,G(\chi_3\phi)\,J(\chi_1\chi_3\phi, \chi_2\chi_3\phi, \overline{\chi_3\phi})}{G(\chi_1\chi_2\chi_3^2)\,\overline{G(\chi_1)}\,\overline{G(\chi_2)}}.$$

Applying (7) with $l = 2$, we find that for any character χ_3,

(26) $$\chi_3(-4)\,G(\chi_3)\,G(\phi)\,G(\chi_3\phi) = \chi_3\phi(-1)\,qG(\chi_3^2).$$

Since χ_1 and χ_2 are nontrivial, it follows from (25) and (26) that

(27) $$S_1 = \frac{\chi_3\phi(-1)\,G(\chi_3^2)\,G(\chi_1)\,G(\chi_2)\,J(\chi_1\chi_3\phi, \chi_2\chi_3\phi, \bar{\chi}_3\phi)}{q\,G(\chi_1\chi_2\chi_3^2)}.$$

Since $\chi_1\chi_2\chi_3\phi$ and $\chi_3\phi$ are nontrivial, (19) now follows from (18) and (27).

Remark. We evaluated S (the left side of (4)) only under the assumption that $\chi_1\chi_2\chi_3^2$ and $(\chi_1\chi_2\chi_3)^2$ were nontrivial. We now indicate how S can be simply evaluated in terms of Gauss sums when this assumption is dropped. If χ_1, χ_2, or χ_3^2 is trivial, one can easily evaluate S directly from its definition. If $\chi_1\chi_2\chi_3^2$ is trivial, then one can evaluate S_1 (and hence S) from (20), by first replacing u by u^{-1}, then replacing v by vu^{-1}, to obtain

$$S_1 = \sum_{u,\,v} \bar{\chi}_1\bar{\chi}_2\bar{\chi}_3^2(u)\,\chi_3(1+u^2+v^2-2u-2v-2uv)\,\chi_2(v)\,.$$

Finally, suppose that χ_1, χ_2, χ_3^2, and $\chi_1\chi_2\chi_3^2$ are nontrivial. Then S_1 can be evaluated from (27).

7. Proof of (5)

Let E denote the left side of (5). Since $\chi_1\chi_2$ is nontrivial,

$$E + 1 + \chi_1\chi_2(-1) = \sum_{\substack{x,\,y\neq 0\\ x+y\neq -1}} \chi_1\chi_3\left(\frac{1+x}{y}\right)\chi_2\chi_3\left(\frac{1+y}{x}\right)\chi_1\chi_2(y-x)\,.$$

Set $t = \dfrac{1+x}{y}$, $\quad u = \dfrac{1+y}{x}$, so

$$x = \frac{t+1}{ut-1}, \qquad y = \frac{u+1}{ut-1}.$$

Then

$$\begin{aligned} E + 1 + \chi_1\chi_2(-1) &= \sum_{\substack{u,\,t\neq -1\\ ut\neq 1}} \chi_1\chi_3(t)\,\chi_2\chi_3(u)\,\chi_1\chi_2\left(\frac{t-u}{1-ut}\right)\\ &= \sum_{u,\,t\neq -1} \chi_1\chi_3(t)\,\chi_2\chi_3(u)\,\chi_1\chi_2(t-u)\,\bar{\chi}_1\bar{\chi}_2(1-ut)\,. \end{aligned}$$

Since $\chi_1\chi_3$ and $\chi_2\chi_3$ are nontrivial,

$$E = \sum_{u,\,t\neq 0} \chi_1\chi_3(t)\,\chi_2\chi_3(u)\,\chi_1\chi_2(t-u)\,\bar{\chi}_1\bar{\chi}_2(1-ut)\,.$$

Replace t by t/u to obtain

$$\begin{aligned} E &= \sum_{u,\,t\neq 0} \chi_1\chi_3(t)\,\bar{\chi}_1^2(u)\,\chi_1\chi_2(t-u^2)\,\bar{\chi}_1\bar{\chi}_2(1-t)\\ &= \sum_{u,\,t\neq 0} \chi_1\chi_3(t)\,\bar{\chi}_1\bar{\chi}_2(1-t)\,\bar{\chi}_1(u)\,\chi_1\chi_2(t-u)\,\{1+\phi(u)\}\,. \end{aligned}$$