

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 27 (1981)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: IDENTITIES FOR PRODUCTS OF GAUSS SUMS OVER FINITE FIELDS
Autor: Evans, Ronald J.
Kapitel: 6. Proof of (4)
DOI: <https://doi.org/10.5169/seals-51748>

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6. PROOF OF (4)

For characters ψ_1, \dots, ψ_m on $GF(q)$, define the Jacobi sum

$$J(\psi_1, \dots, \psi_m) = (-1)^{m+1} \sum_{\substack{x_1, \dots, x_m \in GF(q) \\ x_1 + \dots + x_m = 1}} \psi_1(x_1) \dots \psi_m(x_m).$$

We will use the well-known fact that if $\psi_1 \psi_2 \dots \psi_m$ is nontrivial, then

$$(18) \quad J(\psi_1, \dots, \psi_m) = G(\psi_1 \dots \psi_m)^{-1} \prod_{i=1}^m G(\psi_i).$$

Let S denote the left side of (4). If χ_1, χ_2 , or χ_3^2 is trivial, then it is easy to verify (4) directly, with use of (18) and (26) below. Thus assume that χ_1, χ_2 , and χ_3^2 are nontrivial. With the change of variables

$$u = xy, \quad v = x + y,$$

we have

$$S = \sum_{u, v \in GF(q)} \chi_1(u) \chi_2(1+u-v) \chi_3(v^2-4u) \{1 + \phi(v^2-4u)\}.$$

It therefore remains to show that

$$(19) \quad S_1 = \sum_{u, v} \chi_1(u) \chi_2(1+u-v) \chi_3(v^2-4u) = R(\chi_1, \chi_2, \chi_3 \phi).$$

Replace v by $u + 1 - v$ to get

$$(20) \quad S_1 = \sum_{u, v} \chi_1(u) \chi_2(v) \chi_3(1+u^2+v^2-2u-2v-2uv).$$

Replace u by u/t , and v by v/t , to get

$$(21) \quad \begin{aligned} S_2 &= -S_1 G(\chi_1 \chi_2 \chi_3^2) \\ &= \sum_{t \neq 0} \sum_{u, v} \chi_1(u) \chi_2(v) \chi_3(t^2+u^2+v^2-2ut-2vt-2uv) \zeta^{T(t)}. \end{aligned}$$

Since $\chi_1 \chi_2 \chi_3^2$ is nontrivial, the restriction $t \neq 0$ may be dropped. Then replace t by $t + u + v$ to get

$$S_2 = \sum_{t, u, v} \chi_1(u) \chi_2(v) \chi_3(t^2-4uv) \zeta^{T(t+u+v)}.$$

Replace u by ua and v by vb to get

$$(22) \quad \begin{aligned} S_3 &= S_2 \overline{G(\chi_1)} \overline{G(\chi_2)} \\ &= \sum_t \sum_{a, b, u, v \neq 0} \chi_1(u) \chi_2(v) \chi_3(t^2-4uvab) \zeta^{T(t+a(u-1)+b(v-1))}. \end{aligned}$$

Replace a by $a/(4uvb)$ to get

$$S_3 = \sum_t \sum_{a, b, u, v \neq 0} \chi_1(u) \chi_2(v) \chi_3(t^2 - a) \zeta^{T(t+b(v-1) + \frac{a(u-1)}{4uvb})}$$

Since χ_1 is nontrivial, the restriction $a \neq 0$ may be dropped. Then replace a by $t^2 - a$ to get

$$\begin{aligned} S_3 &= \sum_{a, t} \sum_{b, u, v \neq 0} \chi_1(u) \chi_2(v) \chi_3(a) \zeta^{T(t+b(v-1) + \frac{(1-u)(a-t^2)}{4uvb})} \\ &= -G(\chi_3) \sum_{u \neq 0, 1} \sum_{b, v \neq 0} \chi_1(u) \chi_2(v) \chi_3\left(\frac{4uvb}{1-u}\right) \zeta^{T(b(v-1))} \sum_t \zeta^{T(t + \frac{t^2(u-1)}{4uvb})} \end{aligned}$$

The inner sum on t equals $-\zeta^{T(\frac{uvb}{1-u})} \phi\left(\frac{4uvb}{u-1}\right) G(\phi)$.

Hence

$$\begin{aligned} (23) \quad S_4 &= S_3 (G(\chi_3) G(\phi) \chi_3(-1))^{-1} \\ &= \sum_{u \neq 0, 1} \sum_{b, v \neq 0} \chi_1(u) \chi_2(v) \chi_3 \phi\left(\frac{4uvb}{u-1}\right) \zeta^{T(b(v-1) + \frac{uvb}{1-u})} \end{aligned}$$

Therefore,

$$S_4 = \sum_{u \neq 0, 1} \sum_{v \neq 0} \chi_1 \chi_3 \phi(u) \chi_2 \chi_3 \phi(v) \bar{\chi}_3 \phi\left(\frac{u-1}{4}\right) \sum_b \chi_3 \phi(b) \zeta^{T(b(v-1) + \frac{buv}{1-u})}$$

Since χ_2 and $\chi_3 \phi$ are nontrivial,

$$S_4 = -G(\chi_3 \phi) \sum_{u, v} \chi_1 \chi_3 \phi(u) \chi_2 \chi_3 \phi(v) \bar{\chi}_3 \phi\left(\frac{1-u-v}{4}\right),$$

so

$$(24) \quad S_4 = -\chi_3(4) G(\chi_3 \phi) J(\chi_1 \chi_3 \phi, \chi_2 \chi_3 \phi, \bar{\chi}_3 \phi)$$

Combining (21)-(24), we get

$$(25) \quad S_1 = \frac{\chi_3(-4) G(\chi_3) G(\phi) G(\chi_3 \phi) J(\chi_1 \chi_3 \phi, \chi_2 \chi_3 \phi, \bar{\chi}_3 \phi)}{G(\chi_1 \chi_2 \chi_3^2) G(\chi_1) G(\chi_2)}$$

Applying (7) with $l = 2$, we find that for any character χ_3 ,

$$(26) \quad \chi_3(-4) G(\chi_3) G(\phi) G(\chi_3 \phi) = \chi_3 \phi(-1) q G(\chi_3^2)$$

Since χ_1 and χ_2 are nontrivial, it follows from (25) and (26) that

$$(27) \quad S_1 = \frac{\chi_3 \phi(-1) G(\chi_3^2) G(\chi_1) G(\chi_2) J(\chi_1 \chi_3 \phi, \chi_2 \chi_3 \phi, \bar{\chi}_3 \phi)}{q G(\chi_1 \chi_2 \chi_3^2)}$$

Since $\chi_1\chi_2\chi_3\phi$ and $\chi_3\phi$ are nontrivial, (19) now follows from (18) and (27).

Remark. We evaluated S (the left side of (4)) only under the assumption that $\chi_1\chi_2\chi_3^2$ and $(\chi_1\chi_2\chi_3)^2$ were nontrivial. We now indicate how S can be simply evaluated in terms of Gauss sums when this assumption is dropped. If χ_1 , χ_2 , or χ_3^2 is trivial, one can easily evaluate S directly from its definition. If $\chi_1\chi_2\chi_3^2$ is trivial, then one can evaluate S_1 (and hence S) from (20), by first replacing u by u^{-1} , then replacing v by vu^{-1} , to obtain

$$S_1 = \sum_{u,v} \bar{\chi}_1 \bar{\chi}_2 \bar{\chi}_3^2 (u) \chi_3 (1+u^2+v^2-2u-2v-2uv) \chi_2 (v).$$

Finally, suppose that χ_1 , χ_2 , χ_3^2 , and $\chi_1\chi_2\chi_3^2$ are nontrivial. Then S_1 can be evaluated from (27).

7. PROOF OF (5)

Let E denote the left side of (5). Since $\chi_1\chi_2$ is nontrivial,

$$E + 1 + \chi_1\chi_2(-1) = \sum_{\substack{x,y \neq 0 \\ x+y \neq -1}} \chi_1\chi_3 \left(\frac{1+x}{y} \right) \chi_2\chi_3 \left(\frac{1+y}{x} \right) \chi_1\chi_2 (y-x).$$

Set $t = \frac{1+x}{y}$, $u = \frac{1+y}{x}$, so

$$x = \frac{t+1}{ut-1}, \quad y = \frac{u+1}{ut-1}.$$

Then

$$\begin{aligned} E + 1 + \chi_1\chi_2(-1) &= \sum_{\substack{u,t \neq -1 \\ ut \neq 1}} \chi_1\chi_3(t) \chi_2\chi_3(u) \chi_1\chi_2 \left(\frac{t-u}{1-ut} \right) \\ &= \sum_{u,t \neq -1} \chi_1\chi_3(t) \chi_2\chi_3(u) \chi_1\chi_2(t-u) \bar{\chi}_1 \bar{\chi}_2 (1-ut). \end{aligned}$$

Since $\chi_1\chi_3$ and $\chi_2\chi_3$ are nontrivial,

$$E = \sum_{u,t \neq 0} \chi_1\chi_3(t) \chi_2\chi_3(u) \chi_1\chi_2(t-u) \bar{\chi}_1 \bar{\chi}_2 (1-ut).$$

Replace t by t/u to obtain

$$\begin{aligned} E &= \sum_{u,t \neq 0} \chi_1\chi_3(t) \bar{\chi}_1^2(u) \chi_1\chi_2(t-u^2) \bar{\chi}_1 \bar{\chi}_2 (1-t) \\ &= \sum_{u,t \neq 0} \chi_1\chi_3(t) \bar{\chi}_1 \bar{\chi}_2 (1-t) \bar{\chi}_1(u) \chi_1\chi_2(t-u) \{1 + \phi(u)\}. \end{aligned}$$