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Now replace u by tu to get

$$(28) \quad E = \sum_{u,t} \chi_1 \chi_2 \chi_3 (t) \bar{\chi}_1 \bar{\chi}_2 (1-t) \bar{\chi}_1 (u) \chi_1 \chi_2 (1-u) \{1 + \phi (ut)\} \\ = J (\chi_1 \chi_2 \chi_3, \bar{\chi}_1 \bar{\chi}_2) J (\bar{\chi}_1, \chi_1 \chi_2) + J (\chi_1 \chi_2 \chi_3 \phi, \bar{\chi}_1 \bar{\chi}_2) J (\bar{\chi}_1 \phi, \chi_1 \chi_2),$$

where the Jacobi sums J are defined above (18). Since $\chi_1^2, \chi_2^2, \chi_3^2$, and $\chi_1 \chi_2$ are nontrivial, (5) now follows from (18).

Remark. If $\chi_1 \chi_2, \chi_1 \chi_3$, or $\chi_2 \chi_3$ is trivial, we can easily evaluate E directly from its definition. Otherwise, E can be evaluated simply from (28).

8. CHARACTER SUM ANALOGUES OF (1), (1a) AND (1b)

Let $\chi_1, \chi_2, \chi_3, \phi$ be characters on $GF(q)$, where ϕ has order 2, $p > 2$. Set $t_0 = 1$. The discriminant of the polynomial

$$F(y) = \sum_{i=0}^n t_i y^{n-i}$$

is a polynomial in t_1, \dots, t_n which shall be denoted by D_n . Write

$$E_n = \sum_{i=0}^n (-1)^i t_i.$$

We conjecture that the following analogues of (1), (1a), (1b) hold for each $n \geq 1$:

$$(29) \quad \sum_{t_1, \dots, t_n \in GF(q)} \chi_1 (t_n) \chi_2 (E_n) \chi_3 \phi (D_n) = \prod_{j=0}^{n-1} \frac{-G(\chi_3^{j+1}) G(\chi_1 \chi_3^j) G(\chi_2 \chi_3^j)}{G(\chi_3) G(\chi_1 \chi_2 \chi_3^{n+j-1})},$$

provided that the n characters $\chi_1 \chi_2 \chi_3^{n+j-1}$ ($0 \leq j \leq n-1$) are all nontrivial;

$$(29a) \quad \sum_{t_1, \dots, t_n \in GF(q)} \chi_1 (t_n) \chi_3 \phi (D_n) \zeta^{T(t_1)} = \prod_{j=0}^{n-1} \frac{-G(\chi_3^{j+1}) G(\chi_1 \chi_3^j)}{G(\chi_3)}$$

for all χ_1, χ_3 ; and

$$(29b) \quad \sum_{t_1, \dots, t_n \in GF(q)} \chi_3 \phi (D_n) \zeta^{\frac{p+1}{2} T(t_1^2 - 2t_2)} = \prod_{j=0}^{n-1} \frac{-\phi(2) G(\phi) G(\chi_3^{j+1})}{G(\chi_3)}$$

for all χ_3 .

Formulas (29), (29a), and (29b) have been verified by computer for some small primes q with $n = 3, 4$. Of course the formulas are well known for $n = 1$. For

$n = 2$, (29a) and (29b) are not hard to prove, and (29) follows from the proof of (4). (For example, if χ_3 in (29) is trivial, one makes use of (19) and (21)-(23).)

For $n = 3$, we can prove (29b), but not (29) or (29a). The *ad hoc* proof given below appears to shed little light on the general case.

THEOREM. For $n = 3$, (29b) holds.

Proof. All rational fractions below are to be interpreted as integers (mod p); for example, $\frac{1}{2}$ equals $(p+1)/2$. We must show that

$$(30) \quad A = \sum_{t, u, v} \chi \phi(D_3) \zeta^T\left(\frac{v^2}{2} - u\right) = - \frac{\phi(2) G^3(\phi) G(\chi^3) G(\chi^2)}{G^2(\chi)}$$

for any character χ on $GF(q)$, where

$$D_3 = u^2v^2 - 4u^3 - 4tv^3 - 27t^2 + 18tuv.$$

First suppose that $p = 3$. Then

$$\begin{aligned} A &= \sum_{t, u, v} \chi \phi(u^2v^2 - u^3 - tv^3) \zeta^T\left(\frac{v^2}{2} - u\right) \\ &= \sum_{v \neq 0} \sum_u \zeta^T\left(\frac{v^2}{2} - u\right) \sum_t \chi \phi(u^2v^2 - u^3 - t) + \sum_{t, u} \phi \chi^3(-u) \zeta^T(-u) \\ &= 0 - q G(\phi \chi^3) = -q G(\chi \phi), \end{aligned}$$

since $G(\psi^p) = G(\psi)$ for any character ψ . Now (30) follows with the aid of (26).

Next, suppose that $p > 3$. Completing the square in t , one has

$$D_3/27 = c - \left(t + \frac{2v^3}{27} - \frac{uv}{3}\right)^2,$$

where $c = \frac{4}{27} \left(\frac{v^2}{3} - u\right)^3$. Thus

$$\begin{aligned} \bar{\chi} \phi(27) A &= \sum_{t, u, v} \chi \phi(c - t^2) \zeta^T\left(\frac{v^2}{2} - u\right) \\ &= \sum_{u, v} \zeta^T\left(\frac{v^2}{2} - u\right) \sum_t \chi \phi(c - t) \cdot \{1 + \phi(t)\} \\ &= \sum_{u, v} \zeta^T\left(\frac{v^2}{2} - u\right) \sum_t \chi \phi(c - t) \phi(t) \\ &= K \sum_{\substack{u, v \\ c=0}} \zeta^T\left(\frac{v^2}{2} - u\right) + J \sum_{u, v} \chi(c) \zeta^T\left(\frac{v^2}{2} - u\right), \end{aligned}$$

where $K = \chi\phi(-1) \sum_t \chi(t)$ and $J = \sum_t \chi\phi(1-t)\phi(t)$.

Thus

$$\bar{\chi}\phi(27)A = K \sum_v \zeta^T\left(\frac{v^2}{6}\right) + \chi\left(\frac{4}{27}\right)J \sum_{u,v} \chi^3\left(\frac{v^2}{3} - u\right) \zeta^T\left(\frac{v^2}{2} - u\right).$$

Replace u by $u + \frac{v^2}{3}$ to obtain

$$\begin{aligned} \bar{\chi}\phi(27)A &= -K\phi(6)G(\phi) + \chi\left(\frac{4}{27}\right)J \sum_{u,v} \chi^3(-u) \zeta^T\left(\frac{v^2}{6} - u\right) \\ &= \phi(6)G(\phi) \left\{ -K + \chi\left(\frac{4}{27}\right)JG(\chi^3) \right\}. \end{aligned}$$

If χ is trivial, then $K = \phi(-1)(q-1)$, $J = -\phi(-1)$, and $G(\chi^3) = 1$, and the desired result (30) follows. If χ is nontrivial, then $K = 0$ and

$$J = -G(\chi\phi)G(\phi)/G(\chi)$$

by (18), and (30) follows with the aid of (26).

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