

# 6. Universality of Linear Programming

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We can attempt to generalise the definition of the above vacuous hierarchy by allowing the number of "alternations" to increase with the number of indeterminates.

Let  $t$  be any polynomial. Define  $t-D^0$  to be the class of  $t$ -computable families. For  $i > 0$  let  $t-D^i$  be the class of families that are defined by some family in  $t-D^{i-1}$  in the sense of Definition 3. Finally  $PD^*$  is the class of all families  $P$  such that for some  $t$

$$P = \{P_i \mid P_i = Q_i \text{ for some } Q \in t-D^{(i)}\}.$$

THEOREM 6.  $PD^* = PD^1$

*Proof.* Similar to previous theorem. □

The above two results should be contrasted with the Boolean case where they still hold formally, but are no longer natural. The above definition of the successive levels  $PD^i$  is only natural if each level is a robust closure class. In Boolean algebra, however,  $PD^i$  is not known to be closed under complementation for any  $i \geq 1$ . Analogues of  $PD^i$  and  $PD^*$  where complementation is allowed at each level of alternation are not known to collapse, and are merely finite versions of the Meyer-Stockmeyer hierarchy, and PSPACE respectively [10].

A simple application of Theorem 5 is in recognising such polynomials as  $\#HG$  as being  $p$ -definable. An intriguing open question is whether  $HG$  itself is  $p$ -definable for each  $F$ . If it is not then  $P \neq NP$  (see Proposition 4 in [13]). If it is then the Meyer-Stockmeyer hierarchy and PSPACE can be simulated within  $p$ -definable families of polynomials.

## 6. UNIVERSALITY OF LINEAR PROGRAMMING

Here we consider a Boolean function family  $LP$  that corresponds to a linear programming problem and show that every  $p$ -computable family is the  $p$ -projection of it. Thus for computing discrete functions in polynomial time a package for  $LP$  for each input size is sufficient and no further programming is required. If we fix certain of the arguments of  $LP_i$  according to the particular function and input size being computed, the package becomes a program for the required function. That  $LP$  is itself  $p$ -computable follows from the recent result of Khachian [8].

The reader should note that several tractable problems in combinatorial optimisation are already known to have linear programming formula-

tions [9]. Our result shows that this is a universal phenomenon. It is related to the result in [3].

We define  $LP_{2n(n+1)}$  to be the following Boolean function of arguments  $\{a_{ij}, b_{ij}, e_i, d_i \mid 1 \leq i, j \leq n\}$ :

$$LP(a_{ij}, b_{ij}, e_i, d_i) = 1$$

if and only if the set of inequalities

$$\sum (\tilde{a}_{ij}x_j - \tilde{b}_{ij}x_j) \geq \tilde{e}_i - \tilde{d}_i$$

has a solution in real numbers, where each number  $\tilde{a}_{ij}, \tilde{b}_{ij}, \tilde{e}_i, \tilde{d}_i$  is 1 or 0 according to whether the corresponding Boolean variable  $a_{ij}, b_{ij}, e_i, d_i$  is 1 or 0.

**THEOREM 7.** *Any  $p$ -computable family  $P$  of Boolean functions is the  $p$ -projection of  $LP$ .*

*Proof.* Consider some  $P_m \in P$  with indeterminates  $y_1, \dots, y_m$ , and a minimal program for it. The latter consists of a sequence of instructions of the form  $v_i \leftarrow v_j \wedge v_k$  and  $v_i \leftarrow v_j \vee v_k$ , where  $1 \leq i \leq C$  and each  $v_n$  with  $n \leq 0$  equals some  $y_r$  or  $\bar{y}_r$ .

For any fixed assignment of truth values to  $y_1, \dots, y_m$  we can define a set  $E_0$  of linear inequalities:

$$E_0 = \{x_r \leq 0 \mid r < 0 \text{ and } v_r \text{ has value } 0\} \\ \cup \{x_r \geq 1 \mid r < 0 \text{ and } v_r \text{ has value } 1\}$$

For each sequence  $v_1, v_2, \dots, v_i$  we define  $E_i$  by induction from  $E_0$ :

$$E_i = \begin{cases} E_{i-1} \cup \{x_j - x_i \geq 0, x_k - x_i \geq 0, x_i + 1 - x_j - x_k \geq 0\} \\ \quad \text{if } v_i \leftarrow v_j \wedge v_k, \\ E_{i-1} \cup \{x_j + x_k - x_i \geq 0, x_i - x_j \geq 0, x_i - x_k \geq 0\} \\ \quad \text{if } v_i \leftarrow v_j \vee v_k \end{cases}$$

*Claim 1.* For any  $i, j$  ( $j < i$ ) every solution of  $E_i$  has  $x_j \leq 0$ , or every solution of  $E_i$  has  $x_j \geq 1$ .

*Proof.* The claim is true for  $E_0$  by definition. Assume inductively that it is true for  $E_{i-1}$ . (a) If  $v_i \leftarrow v_j \wedge v_k$  then  $x_j \leq 0$  implies that  $x_i \leq 0$  since  $x_j - x_i \geq 0$ . Similarly if  $x_k \leq 0$ . In the remaining case  $x_j, x_k \geq 1$  inequality  $x_i + 1 - x_j - x_k \geq 0$  ensures that  $x_i \geq 1$ . (b) If  $v_i \leftarrow v_j \vee v_k$  then  $x_j \geq 1$

implies that  $x_i \geq 1$  since  $x_i - x_j \geq 0$ . Similarly if  $x_k \geq 1$ . If  $x_j, x_k \leq 0$  then  $x_j + x_k - x_i \geq 0$  ensures that  $x_i \leq 0$ .  $\square$

*Claim 2.* If  $\text{val}(v_i) = 0$  then  $E_i \cup \{x_i \leq 0\}$  has a solution. If  $\text{val}(v_i) = 1$  then  $E_i \cup \{x_i \geq 1\}$  has a solution.

*Proof.* By induction on  $i$  it is easy to see that the point

$$x_j = \begin{cases} 1 & \text{if } \text{val}(v_j) = 1 \\ 0 & \text{if } \text{val}(v_j) = 0 \end{cases}$$

for  $1 \leq j \leq i$  is a solution of  $E_i$ .  $\square$

*Claim 3.* If for some  $i, j (j \leq i)$   $E_i \cup \{x_j \geq 1\}$  has a solution in reals then  $\text{val}(v_j) = 1$ .

*Proof.* By Claim 1, if  $E_i \cup \{x_j \geq 1\}$  has a solution then  $E_i \cup \{x_j \leq 0\}$  has no solution. Hence by Claim 2  $\text{val}(v_j) = 1$ .  $\square$

Finally we observe that the given program of size  $C$  for  $P_m$  translates to  $3C + 2m$  inequalities in  $E_C$ , of which the  $2m$  of  $E_o$  depend on the values of  $y_1, \dots, y_m$ , while the remaining  $3C$  are fixed. It remains to note that  $P_m$  is the projection under  $\sigma$  of  $LP_{2n(n+1)}$  for  $n = 3C + 2m$ , where  $\sigma$  maps  $3C$  of the inequalities to those of  $E_C - E_o$ , and the remaining  $2m$  values of  $i$  as follows. If  $v_i$  equals  $y_j$  or  $\bar{y}_j$  then:  $\sigma(a_{ik}) = \sigma(b_{ik}) = 0$  if  $j \neq k$ ,  $\sigma(d_i) = 0$ ,  $\sigma(a_{ij}) = \sigma(e_i) = v_i$ ,  $\sigma(b_{ij}) = \bar{v}_i$ .  $\square$

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### APPENDIX 1

We show here that in the concept of  $p$ -definability it is immaterial whether the defining polynomials allowed are the  $p$ -computable ones or merely those of  $p$ -bounded formula size. We shall suppose that the family  $P$  is  $p$ -definable in the sense of Definition 3, i.e.

$$P_n(x_1, \dots, x_n) = \sum_{b \in \{0,1\}^{m-n}} Q_m(x_1, \dots, x_n, b_{n+1}, \dots, b_m)$$

It will suffice to prove that any  $p$ -computable family, such as  $Q$ , is  $p$ -definable in the sense of Definition 4. By Theorem 5 it then follows that  $P$  itself is also  $p$ -definable in the sense of Definition 4.