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implies that  $x_i \ge 1$  since  $x_i - x_j \ge 0$ . Similarly if  $x_k \ge 1$ . If  $x_j, x_k \le 0$  then  $x_j + x_k - x_i \ge 0$  ensures that  $x_i \le 0$ .

Claim 2. If val  $(v_i) = 0$  then  $E_i \cup \{x_i \le 0\}$  has a solution. If val  $(v_i) = 1$  then  $E_i \cup \{x_i \ge 1\}$  has a solution.

*Proof.* By induction on *i* it is easy to see that the point

$$x_j = \begin{cases} 1 & \text{if val } (v_j) = 1 \\ 0 & \text{if val } (v_j) = 0 \end{cases}$$

for  $1 \leq j \leq i$  is a solution of  $E_i$ .

Claim 3. If for some  $i, j (j \le i) E_i \cup \{x_j \ge 1\}$  has a solution in reals then val  $(v_j) = 1$ .

*Proof.* By Claim 1, if  $E_i \cup \{x_j \ge 1\}$  has a solution then  $E_i \cup \{x_j \le 0\}$  has no solution. Hence by Claim 2 val  $(v_j) = 1$ .

Finally we observe that the given program of size C for  $P_m$  translates to 3C + 2m inequalities in  $E_c$ , of which the 2m of  $E_o$  depend on the values of  $y_1, ..., y_m$ , while the remaining 3C are fixed. It remains to note that  $P_m$ is the projection under  $\sigma$  of  $LP_{2n(n+1)}$  for n = 3C + 2m, where  $\sigma$  maps 3Cof the inequalities to those of  $E_c - E_o$ , and the remaining 2m values of *i* as follows. If  $v_i$  equals  $y_j$  or  $\bar{y}_j$  then:  $\sigma(a_{ik}) = \sigma(b_{ik}) = 0$  if  $j \neq k, \sigma(d_i)$  $= 0, \sigma(a_{ij}) = \sigma(e_i) = v_i, \sigma(b_{ij}) = \bar{v}_i$ .

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# **APPENDIX** 1

We show here that in the concept of p-definability it is immaterial whether the defining polynomials allowed are the p-computable ones or merely those of p-bounded formula size. We shall suppose that the family Pis p-definable in the sense of Definition 3, i.e.

$$P_n(x_1,...,x_n) = \sum_{b \in \{0,1\}} Q_m(x_1,...,x_n,b_{n+1},...,b_m)$$

It will suffice to prove that any *p*-computable family, such as Q, is *p*-definable in the sense of Definition 4. By Theorem 5 it then follows that P itself is also *p*-definable in the sense of Definition 4.

It is known that any *p*-computable family of homogeneous polynomials has homogeneous program size at most polynomially larger than its unrestricted program size [12]. The inductive proof to follow assumes the former measure throughout and supports homogeneity. We shall assume that  $Q_m$  is itself homogeneous. If it were not then we would consider each of its homogeneous components separately in the same way.

Suppose that  $Q_m(x_1, ..., x_m)$  has degree d and a minimal program  $\rho$  of complexity C. Let U be the subset of the computed terms  $\{v_i\}$  such that (i) deg  $(v_i) > d/2$  and (ii)  $v_i \leftarrow v_j \times v_k$  with deg  $(v_j) \leq d/2$  and deg  $(v_k) \leq d/2$ . Let W be the subset  $\{v_j\}$  such that  $v_i \leftarrow v_j \times v_k$  or  $v_j \leftarrow v_k \times v_j$  for some  $v_i \in U$ . For convenience rename the elements of U and W by  $\{u_1, ..., u_r\}$  and  $\{w_1, ..., w_s\}$  respectively.

Claim 1. There is a polynomial  $S_{m+r+1}$   $(x_1, ..., x_m, e_0, ..., e_r)$  of degree  $\lfloor d/2 \rfloor + 1$  and homogeneous program complexity at most 2C + d such that

$$Q_m(\mathbf{x}) = \sum_{i=1}^r \operatorname{val}(u_i) \cdot \operatorname{compl}_i$$

where compl<sub>i</sub> =  $S_{m+r+1}$  (**x**, **e**) when  $e_0 = e_i = 1$  and  $e_j = 0$  for  $0 \neq j \neq i$ .

*Proof.* In  $\rho$  replace each occurrence of  $u_i$  on the right hand side of an assignment by an occurrence of  $e_i e_0^{\deg(u_i) - \lfloor d/2 \rfloor - 1}$ . (Actually this would be simulated by a subprogram that raises  $e_0$  to every power and multiplies by  $e_i$  as appropriate.)

Claim 2. There is a polynomial  $T_{n+s+1}$   $(x_1, ..., x_m, c_0, ..., c_s)$  of degree  $\lfloor d/2 \rfloor + 1$  and homogeneous program complexity at most 3C + d such that for each i  $(1 \le i \le s)$ 

val 
$$(w_i) = T_{m+s+1}(\mathbf{x}, \mathbf{c})$$

when  $c_0 = c_i = 1$  and  $c_j = 0$  for  $0 \neq j \neq i$ .

*Proof.* Delete from  $\rho$  every instruction with degree greater than d/2. Add a subprogram equivalent to the set of instructions

$$Z_i \leftarrow W_i \times c_i c_0^{\lfloor d/2 \rfloor - \deg(w_i)}$$

for i = 1, ..., s. Add further instructions to sum  $z_1, ..., z_s$ .

Now for each *i* val  $(u_i) = val(w_j) val(w_k)$  for some *j*, *k* specified by  $\rho$ . Hence each of the *r* additive contributions to  $Q_m$  is some product

$$T_{m+s+1}(\mathbf{x}, \mathbf{c}) T_{m+s+1}(\mathbf{x}, \mathbf{c}') S_{m+r+1}(\mathbf{x}, \mathbf{e})$$

where  $(\mathbf{c}, \mathbf{c}', \mathbf{e})$  is a fixed (0, 1)-vector of 2s+r+3 elements. But any such vector can be specified by a conjunction of 2s+r+3 Boolean literals. Consider the disjunction of the *r* such conjunctions and let *R* ( $\mathbf{c}, \mathbf{c}', \mathbf{e}$ ) be the polynomial that simulates this Boolean formula at (0, 1) values. Then clearly

$$Q_m(\mathbf{x}) = \sum T(\mathbf{x}, \mathbf{c}) T(\mathbf{x}, \mathbf{c}') S(\mathbf{x}, \mathbf{e}) R(\mathbf{c}, \mathbf{c}', \mathbf{e}) ,$$

where summation is over  $(\mathbf{c}, \mathbf{c}', \mathbf{e}) \in \{0, 1\}^{2s+r+3}$ .

Let A(C, d) be the upper bound over every homogeneous polynomial having degree d and homogeneous program complexity C, of the minimal size of formula needed to define it in Definition 4. Then the above recursive expression ensures that

$$A(C,d) \leqslant 3A(3C+d, \lfloor d/2 \rfloor + 1) + 0(C).$$

Clearly also  $A(C, 1) \leq 2C$ . Hence if d is p-bounded in m then so is the solution to this recurrence.

# Appendix 2

For completeness we describe here a direct proof of the *p*-definability of *HC* in the sense of Definition 1.  $HC_{n \times n}(x_{i,j})$  will be the projection under

 $\sigma(u_{k,m}) = 1 \quad \text{for} \quad 1 \leq k, m \leq n$ 

of the polynomial in  $\{x_{i,j}, u_{k,m}\}$  defined by

$$Q_{n \times n}(y_{i, j}) \cdot Q_{n \times n}(z_{k, m}) \cdot R^1 \dots R^n$$

with the association  $y_{i,j} \leftrightarrow x_{i,j}$  and  $z_{k,m} \leftrightarrow u_{k,m}$ . Here  $Q_{n \times n}$  is the polynomial that defines the permanent in §3. Its first occurrence with argument y plays exactly the same role as in the permanent and ensures a cycle cover. The intention of  $z_{k,m}$  is to denote whether the  $k^{th}$  node in the circuit (starting from node 1, say) is node m.  $Q_{n \times n}(z_{k,m})$  ensures that this intention is realised. For each  $k \ R^k$  captures the fact that if  $z_{k,m}$  and  $z_{k+1,r}$  are both 1 then  $y_{m,r}$  must be also. In Boolean notation we require

$$y_{m, r} \vee (\overline{z}_{k, m} \vee \overline{z}_{k+1, r})$$
.

As is well known such Boolean formulae can be simulated by polynomials at  $\{0, 1\}$  values (e.g. see Proposition 2 in [13]). To guarantee just one monomial for each cycle we fix  $R^1 = z_{11}$ .

L'Enseignement mathém., t. XXVIII, fasc. 3-4.