1. Introduction

HARMONIZABLE PROCESSES: STRUCTURE THEORY [1]

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Dedicated to the memory of Prof. S. Bochner

CONTENTS

1. INTRODUCTION

If $\mathscr{H}$ is a complex Hilbert space and $X : \mathbf{R} \to \mathscr{H}$ is a mapping, then the curve $\{X(t), t \in \mathbf{R}\}$ is often called a *second order* (or Hilbertian) stochastic process, and if $\mathbf{R}$ is replaced by $\mathbf{R}^n$, $n \geqslant 2$, it is called a (Hilbertian) *random field.* Following Khintchine who developed the initial theory (1934), the process (or field) is called *weakly stationary* if $r : (s, t) \mapsto (X(s), X(t))$, termed the *covariance function* of the

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process, is continuous and depends only on $s - t$, where $(\cdot, \cdot)$ is the inner product in $\mathcal{H}$. Thus $r(s, t) = r(s-t)$. But then $r : \mathbf{R} \to \mathbf{C}$ is a continuous positive definite function and by the classical Bochner theorem (1932), r is expressible as:

$$r(t) = \int_{\mathbf{R}} e^{it\lambda} F(d\lambda), \qquad t \in \mathbf{R}, \tag{1}$$

for a unique positive bounded Borel measure F on $\mathbf{R}$. This F is called the *spectral measure* of the process. Because of the above connection with the Fourier transform theory, important advances have been made on the structural analysis of such stationary processes. For instance, according to a celebrated theorem of Cramér and Kolmogorov, each such stationary process admits an integral representation:

$$X(t) = \int_{\mathbf{R}} e^{it\lambda} Z(d\lambda), \qquad t \in \mathbf{R}, \tag{2}$$

where Z is an $\mathcal{H}$-valued "orthogonally scattered" measure on the Borel sets of $\mathbf{R}$ (i.e., Z is σ-additive and $(Z(A), Z(B)) = F(A \cap B)$), and the vector integral in (2) is suitably defined. Stationary processes find important applications in such areas as meteorology, communication and electrical engineering among others. The well developed theory and applications are now included in many monographs (cf. e.g. Doob [6, Ch. X-XII], Yaglom [44]), and especially for applications one may refer to Wiener's pioneering work [43].

While stationary processes (the qualification "weakly" will be dropped) admit a deep and beautiful mathematical theory, there are many problems for which stationarity is an unacceptable restriction. For instance, in econometrics and in the signal detection problems related to the navy, among others, it is quite desirable that the covariance function r be not so restricted as to be a function of a single variable. This necessitates a relaxation of stationarity and then (1) cannot obtain. To accommodate such problems while still retaining the methods of harmonic analysis, Loève has introduced in the middle 1940's the first weakening called "harmonizability". Thus a process $\{X(t), t \in \mathbf{R}\} \subset \mathcal{H}$ is *Loève* (to be called *strongly* hereafter) *harmonizable* if its covariance is expressible as (cf. [23], p. 474)

$$r(s, t) = \int_{\mathbf{R}} \int_{\mathbf{R}} e^{is\lambda - it\lambda'} F(d\lambda, d\lambda'), \qquad s, t \in \mathbf{R}, \tag{3}$$

for a unique positive definite $F : \mathbf{R} \times \mathbf{R} \to \mathbf{C}$ of bounded variation (in the classical Vitali sense) in the plane. If F of (3) concentrates on the diagonal of $\mathbf{R} \times \mathbf{R}$, (3) reduces to (1). Loève also gave a representation of $X(t)$ analogous to (2), but now $Z(\cdot)$ will only satisfy $(Z(A), Z(B)) = F(A, B)$. Even though $r(\cdot, \cdot)$ of (3) is bounded and uniformly continuous, one does not have an elegant characterization of a harmonizable covariance analogous to (1). In fact

Loève raised this problem ([23], p. 477). A solution of it was presented in ([34], Thm. 5), but it is not effective in the sense that the conditions are not easily verifiable, although the characterization reduces to Bochner's theorem in the stationary case.

Other extensions of stationarity, of interest in applications, soon appeared. In 1947, Karhunen introduced a class of processes whose covariance r can be expressed as:

$$r(s, t) = \int_{\mathbf{R}} g(s, \lambda)\bar{g}(t, \lambda)F(d\lambda), \qquad s, t \in \mathbf{R}, \tag{4}$$

where $\{g(t, \cdot), t \in \mathbf{R}\}$ is a family of Borel functions in $L^2(\mathbf{R}, F(d\lambda))$, with F as a bounded (or σ-finite) Borel measure on $\mathbf{R}$. If $g(t, \lambda) = e^{it\lambda}$, then for bounded F (4) reduces to (1). In 1951, Cramér has introduced in [3] a further generalization, to be called *class* (C) here, which contains both (3) and (4), by requiring only that r be representable as:

$$r(s, t) = \int_{\mathbf{R}}\int_{\mathbf{R}} g(s, \lambda)\bar{g}(t, \lambda')F(d\lambda, d\lambda'), \qquad s, t \in \mathbf{R}, \tag{5}$$

for a family $\{g(t, \cdot), t \in \mathbf{R}\}$ of Borel functions and a positive definite F of finite local (i.e., on each relatively compact rectangle) Vitali variation in $\mathbf{R}^2$, such that (5) holds. The corresponding stochastic integral representation of $X(t)$, generalizing (2), was also given. Both (4) and (5) have only a superficial contact with the methods of Fourier analysis. However, a very general concept which fully utilizes the advantages of Fourier analysis and which contains the Loève harmonizability was introduced by Bochner in 1953 under the name V-boundedness [2]. It turns out that (cf. Thm. 4.2 below) a second order process is V-bounded iff (= if and only if) it is the Fourier transform of a general vector measure on $\mathbf{R}$ into a Banach space $\mathcal{X}$. Independently of the work of [2], Rozanov [40], in 1959, considered a generalized concept again under the name "harmonizable", but which is different from Loève's definition. It will be called *weakly harmonizable* here. It turns out that, in this case, the covariance function r of the process is formally expressible in the form (3) relative to a positive definite F which is merely of Fréchet variation finite. The integral in (3) then cannot be defined in the Lebesgue sense, and a weaker Morse-Transue integral [26] appears in this work.

Even though each of these generalizations is inspired by the stationarity notion of Khintchine, each is different from one another, and their interrelations have not been fully established before. One of the main purposes of this paper is to present a detailed and unified structural analysis of these processes and obtain their characterization. This exposition utilizes some elementary aspects of vector measure theory which obviates a separate definition of the "stochastic integral" for each representation of the process under consideration in the form (2). From this analysis one finds that Loève's definition is more restrictive than Rozanov's and that Bochner's concept is mathematically the most elegant and general. Further in the Hilbert space context, it is shown that Bochner's and Rozanov's concepts coincide. It was already noted in [2] that Loève's definition is subsumed by V-boundedness. An interesting geometrical feature is that the Bochner class of second order processes is always a projection of a stationary family in a Hilbert space. Bochner's concept, as indicated above, is based on Fourier vector integration, and this abstract point of view yields different characterizations, one of which extends a scalar result of Helson [12] on characterizing Fourier transforms of signed measures, to separable reflexive Banach spaces. A further relation is that a process of the Bochner-Rozanov class in Hilbert space is a strong limit of a sequence of Loève harmonizable processes, uniformly on compact subsets of the line **R**.

A first comparative study of the Bochner and Loève classes in Hilbert space was given by Niemi in his thesis [29]. Then in [30] and [31] he essentially established that the V-boundedness in Hilbert space is the projection of a stationary family, extending a special case by Abreu [1]. The latter point was clarified and the same result was reestablished by a slightly different method in [25]. A further extension of the last work was announced in [39]. A key domination inequality, on which the projection results depend, is based on some work of Grothendieck. In particular, the methods of [25], [30] and [31] rest on Pietsch's form of this Grothendieck inequality. The work of the present paper utilizes some properties of the p-summing operators of [22]. I believe that the latter point of view yields a better understanding of the structure of the problem, with a more general solution and additional insight, not afforded by the earlier work. Thus the present paper is aimed at a comprehensive, unified and extended treatment of the structure of the Bochner-Rozanov class. It may be remarked that an essentially equivalent characterization of Bochner's Hilbert space version can be obtained using the results from an early paper due to Phillips [33], which seems to have been overlooked by almost all vector measure theorists and stochastic analysts. It is, in a sense, subsumed under a relatively recent paper by Kluvanek [21]. But most of all, Bochner's paper [2] has not been accorded the central place it deserves in probabilistic treatments on the subject. I hope that the present work will bring some of the many fundamental ideas of [2] to the forefront.

Finally, the concept of the spectral measure F of (1), so appropriate and natural in the stationary case (since it is *positive* and bounded) does not appear in a similar form for the harmonizable (or other nonstationary) processes, since F is usually complex valued as in (3) or (5). To overcome this problem, in the late 1950's, Kampé de Fériet and Frenkiel ([15], [16]) and independently Parzen [32] and Rozanov [40] have defined an "associated spectrum" for a class of second order processes $X : \mathbf{R} \to L_0^2(P)$. These are processes for which

$$\lim_{T \to \infty} \frac{1}{T} \int_0^{T-|h|} (X(s), X(s+|h|))ds = \tilde{r}(h), \quad h \in \mathbf{R}, \tag{6}$$

exists. Since $\tilde{r}(\cdot)$ is clearly positive definite, one can apply the Bochner representation theorem as in (1), in many cases. The resulting positive bounded measure F for this $\tilde{r}$ is called the *associated spectrum* of the process X. This class, to be termed *class* (KF), contains not only stationary processes but, among others, many almost periodic ones [35]. With the present methods it is shown in Section 8 that every weakly harmonizable process has an associated spectrum from which in fact several other properties can be obtained. A distinguishing feature of the weakly harmonizable case from the stationary, Cramér, Karhunen, or Loève definitions is that the theory of bimeasures and the consequent (nonabsolute) integration of Morse and Transue ([26], [27], [42], [45], [46]) play a vital role in their analyses. This difference has not been fully appreciated in the literature. (The most comprehensive characterizations of the harmonizable class are summarized in Theorems 7.3 and 7.4.) For vector valued processes, in both the weak and strong cases, some new technical problems have to be resolved. The same is true of random fields. All these aspects have important applications and some indications are given in Sections 8 and 9. A summary of some of these results is included in [37]. For greater accessibility and convenience, the next three sections are devoted to harmonizable processes and most of the remaining five consider the more general random fields with a natural transition. However, an essentially self-contained exposition (modulo some standard measure theory) is presented here.

Notation: The following notation is used: $\mathbf{R}$ for reals, $\mathbf{C}$ for complex numbers, $\mathbf{Z}$ for integers, $\mathbf{R}^n$ for the n-dimensional number space, and LCA for locally compact abelian. A *step function* is a mapping taking finitely many values on disjoint measurable sets, and a *simple function* on a measure space is a step function vanishing outside of a set of finite measure. Overbar denotes complex conjugation.