2. Harmonizability

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2. HARMONIZABILITY

For the work of this paper it is convenient to take the Hilbert space $\mathscr{H}$ as the standard function space. Namely, let (Ω, Σ, P) be a probability space and

$$L^2(P)\left(= L^2(\Omega, \Sigma, P)\right)$$

be the space of (equivalence classes of) scalar square integrable functions (= random variables) on Ω, and set

$$\mathscr{H} = L_0^2(P) = \{f \in L^2(P) : \int_\Omega f dP = 0\}.$$

This choice does not really restrict the generality since any abstract Hilbert space is known to be realizable isometrically as a subspace of $L^2(P)$ on some probability space (cf. e.g., [36], p. 414). From this point of view, a process $\{X(t), t \in \mathbf{R}\} \subset L_0^2(P)$ is stationary if its covariance r satisfies $r(s, t) = r(s-t)$, where

$$r(s, t) = E\left(X(s)\overline{X}(t)\right) = \int_\Omega X(s)\overline{X}(t)dP = \left(X(s), X(t)\right), \qquad s, t \in \mathbf{R},$$

and E is also called the "expectation" (= integral). Since $r(\cdot)$ is of positive type (= positive [semi-] definite), assuming it to be jointly measurable (this is implied by the measurability of the random function $\{X(t), t \in \mathbf{R}\}$), it follows that r admits the representation

$$r(t) = \int_{\mathbf{R}} e^{it\lambda} F(d\lambda), \qquad a.a.(t) \quad \text{(Leb.)}. \tag{1'}$$

It may be remarked that in the original (1932) version, Bochner assumed that $r(\cdot)$ is actually continuous, but soon afterward in (1933) F. Riesz showed that measurability itself yields this (slightly weaker) form (1'). This was also used in [33].

For a stationary process $\{X(t), t \in \mathbf{R}\}$, one easily verifies that it is mean continuous (i.e., $E(|X(s) - X(t)|^2) \to 0$ as $s \to t$) iff the covariance $r(\cdot, \cdot)$ is continuous on the diagonal of $\mathbf{R} \times \mathbf{R}$. Thus the measurability of r and the validity of (1') everywhere implies already the mean continuity of the stationary process! So for certain applications of the type noted earlier, it is desirable to weaken the hypothesis of stationarity retaining some representative features. This was done by Loève, and it is restated in the following form:

Definition 2.1. A process $X : \mathbf{R} \to L_0^2(P)$ is *strongly harmonizable* if its covariance r is the Fourier transform of some covariance function ρ of bounded variation, so that one has

$$r(s,t) = \int_{\mathbf{R}}\int_{\mathbf{R}} e^{is\lambda - it\lambda'} \rho(d\lambda, d\lambda'), \qquad s, t \in \mathbf{R}. \tag{3'}$$

It was noted in the Introduction that there is no efficient characterization of r given by (3′). There is however a more visible drawback of this concept. Since strong harmonizability is derived from stationarity, so that the latter class is included, consider a "truncated series" $\{\tilde{X}(n), n \in \mathbf{Z}\}$ of a stationary series $\{X(n), n \in \mathbf{Z}\}$ defined as: $\tilde{X}(n) = X(n)$ for finitely many n, and $\tilde{X}(n) = 0$ for all other $n \in \mathbf{Z}$. Then $\{\tilde{X}(n), n \in \mathbf{Z}\}$ is easily seen to be strongly harmonizable. But if $\tilde{X}(n) = X(n)$, for infinitely many n, and $= 0$ for all other n, then $\{\tilde{X}(n), n \in \mathbf{Z}\}$ need not be strongly harmonizable, as the following example illustrates.

Let (Ω, Σ, P) be separable and $\{f_n, n \in \mathbf{Z}\} \subset L_0^2(P)$ be a complete orthonormal set. Then $r(m, n) = \delta_{m-n} = r(m-n)$. So the sequence is stationary and (1′) becomes

$$r(m-n) = \int_{-\pi}^{\pi} e^{i(m-n)\lambda} \frac{d\lambda}{2\pi}, \qquad m, n \in \mathbf{Z}.$$

Now consider the truncated series, $\tilde{f}_n = f_n$, $n > 0$, and $= 0$ for $n \leqslant 0$. Then $\tilde{r}(m, n) = E(\tilde{f}_m \bar{\tilde{f}}_n) = 1$ if $m = n > 0$, $= 0$ otherwise. But $\tilde{r}$ does not admit the representation (3′). For, otherwise, $\tilde{r}(m, n)$ will be the Fourier coefficient of the representing ρ (of bounded variation) which is only nonvanishing on the ray $(m = n > 0)$ in $\mathbf{Z}^2$. It is a consequence of an important two dimensional extension by Bochner of the classical F. and M. Riesz theorem that ρ must then be absolutely continuous relative to the planar Lebesgue measure with density ρ'. But this implies $\tilde{r}(m, n) \to 0$ as $|m| + |n| \to \infty$ by the Riemann-Lebesgue lemma, and contradicts the fact that $\tilde{r}(m, n) = 1$, for all positive $m = n$ and $n \to \infty$. Hence $\tilde{r}$ cannot admit the representation (3′) so that $\{\tilde{f}_n, n \in \mathbf{Z}\}$ is *not* strongly harmonizable. This example is a slight modification of one due to Helson and Lowdenslager ([13], p. 183) who considered it for a similar purpose, and also appears in [1] for a related elucidation.

The above example and discussion lead us to look for a weakening of the conditions on the covariance function, since it is reasonable to expect *each* truncation of a stationary series to be included in a generalization, retaining the other properties as far as possible. Such an extension was successfully obtained in two different forms in the works of Bochner [2] and Rozanov [40]. The precise concept can be stated and its significance appreciated only after some preliminary considerations.

The measure function ρ of (3′) has the following properties:

(i) ρ is positive definite, i.e.

$$\rho(s, t) = \overline{\rho(t, s)}, \sum_{i=1}^{n} \sum_{j=1}^{n} a_i \overline{a_j}\, \rho(s_i, s_j) \geqslant 0, \qquad a_i \in \mathbf{C},$$

(ii) ρ is of bounded variation, i.e.

$$\sup \{\sum_{i=1}^{n} \sum_{j=1}^{n} \int_{A_i} \int_{B_j} |\rho(ds, dt)| : A_i, B_j \in \mathscr{B}, \text{ disjoint}\} < \infty,$$

where $\mathscr{B}$ is the Borel σ-algebra of $\mathbf{R}$. If $F : \mathscr{B} \times \mathscr{B} \to \mathbf{C}$ is defined by $F(A, B) = \int_A \int_B \rho(ds, dt)$, it follows from (i) and (ii) that there exists a complex Radon measure μ on $\mathbf{R}^2$ such that $F(A, B) = \mu(A \times B)$, where $A \times B \in \mathscr{B} \otimes \mathscr{B}$, and μ is positive definite. On the other hand, the defining equation of F implies that F is positive definite (so (i) holds with $\rho(s_i, s_j)$ replaced by $F(A_i, A_j)$) and (ii) becomes

$$V(F) = \sup \{\sum_{i=1}^{n} \sum_{j=1}^{n} |F(A_i, B_j)| : A_i, B_j \in \mathscr{B}, \text{ disjoint}\} < \infty.$$

But (3′) is meaningful, if ρ is replaced by F under the following weaker conditions.

Let $F : \mathscr{B} \times \mathscr{B} \to \mathbf{C}$ be positive definite and be σ-additive in each variable separately. Equivalently, if $\mathscr{M}(\mathbf{R}, \mathscr{B})$ is the vector space of complex measures on $\mathscr{B}$, let $\nu(A) = F(A, \cdot)$, $A \in \mathscr{B}$ so that $\nu : \mathscr{B} \to \mathscr{M}(\mathbf{R}, \mathscr{B})$ is a vector measure. By symmetry, $\tilde{\nu} : B \to F(\cdot, B)$ is also a vector measure on $\mathscr{B} \to \mathscr{M}(\mathbf{R}, \mathscr{B})$. But $\mathscr{M}(\mathbf{R}, \mathscr{B}) = \mathscr{X}$ is a Banach space under the total variation norm, and hence ν (as well as $\tilde{\nu}$) has *finite semivariation* by a classical result (cf. [8], IV.10.4). This means,

$$\| \nu \| (\mathbf{R}) = \sup \{\| \sum_{i=1}^{n} a_i \nu(A_i) \|_{\mathscr{X}} : |a_i| \leqslant 1, A_i \in \mathscr{B}, \text{ disjoint}\} < \infty.$$

Transferred to F, this translates to:

$$\| F \| (\mathbf{R} \times \mathbf{R}) = \sup \{\sum_{i=1}^{n} \sum_{j=1}^{n} a_i \overline{a_j}\, F(A_i, A_j) : A_i \in \mathscr{B}, \text{ disjoint}, |a_i| \leqslant 1\} < \infty. \tag{7}$$

When (7) holds, $F : \mathscr{B} \times \mathscr{B} \to \mathbf{C}$ will be called a **C**-*bimeasure of finite semivariation.* [It should be noted that the σ-additivity of $F(\cdot, \cdot)$ in each of its components can be replaced by finite additivity and continuity of F from above at $\emptyset$ in that $|F(A_n, A_n)| \to 0$ as $A_n \downarrow \emptyset$.] The desired generalization follows from (7) if it is written in the following form. Let $\varphi = \sum_{i=1}^{n} a_i \chi_{A_i}$ and

$$\psi = \sum_{j=1}^{n} b_j \chi_{B_j}, \; A_i \in \mathscr{B}, \; B_j \in \mathscr{B}$$

and each collection is disjoint. Set

$$I(\varphi, \psi) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i \bar{b}_j F(A_i, B_j) . \tag{8}$$

Clearly I is well-defined, does not depend on the representation of φ or ψ, and $I(\varphi, \varphi) \geqslant 0$. So $(\varphi, \psi) = I(\varphi, \psi)$ is a semi-inner product on the space of $\mathscr{B}$-step functions. Hence by the generalized Schwarz's inequality one has:

$$| I(\varphi, \psi) |^2 \leqslant I(\varphi, \varphi) \cdot I(\psi, \psi) . \tag{9}$$

Taking suprema on all such step functions φ, ψ such that

$$\| \varphi \|_u \leqslant 1, \| \psi \|_u \leqslant 1$$

($\| \cdot \|_u$ is the uniform norm), one deduces from (9) and (7) that

$$\| F \| (\mathbf{R} \times \mathbf{R}) \leqslant \sup \{| \sum_{i=1}^{n} \sum_{j=1}^{n} a_i \bar{b}_j F(A_i, B_j) | : | a_i | \leqslant 1,$$

$$| b_j | \leqslant 1, A_i, B_j \in \mathscr{B}, \text{disjoint}\} \leqslant \| F \| (\mathbf{R} \times \mathbf{R}), (\leqslant V(F)) . \tag{10}$$

Thus $\| F \| (\mathbf{R} \times \mathbf{R})$ can be defined either by the middle term (as in [40]) or by (7). For a bimeasure, $\| F \| (\mathbf{R} \times \mathbf{R})$ is also called *Fréchet variation* of F (cf. [26], p. 292) and $V(F)$ the *Vitali variation*, (cf. [26], p. 298).

It should be emphasized that a set function F which is only a bimeasure (even positive definite), need *not* define a (complex) Radon measure on $\mathbf{R}^2$. In fact such bimeasures do not necessarily admit the Jordan decomposition, as counter examples show. Thus integrals relative to F (even if $\| F \| (\mathbf{R} \times \mathbf{R}) < \infty$) *cannot* generally be of Lebesgue-Stieltjes type. Treating $\nu: A \mapsto F(A, \cdot)$, $A \in \mathscr{B}$, as a vector measure into $\mathscr{M}(\mathbf{R}, \mathscr{B})$, one can employ the Dunford-Schwartz *(or D-S) integral* (cf. [8], IV.10), or alternately one can use the theory of bimeasures as developed in ([26], [27]) and [42]. This is the price paid to get the desired weakened concept, but it will be seen that a satisfactory solution of our problem is then obtained, and both these integrations will play key roles.

Let us therefore recall an appropriate integration concept to be used in the following. In ([40], p. 276) Rozanov has indicated a modification without detailing the consequences. (This resulted in a conjecture [40, p. 283] which will be resolved in Section 8 below.) Instead, a *different* route will be followed: namely the integration theory of Morse and Transue will be used from [27] together with a related result of Thomas ([42], p. 146). However, the Bourbaki set up of these papers is inconvenient here, and they will be converted to the set theoretical (or ensemble) versions and employed.

Let $F : \mathscr{B} \times \mathscr{B} \to \mathbf{C}$ be a bimeasure, i.e. $F(\cdot, B)$, $F(A, \cdot)$ are complex measures on $\mathscr{B}$. Hence one can define as usual ([8], III.6),

$$\tilde{I}_1(f, A) = \int_{\mathbf{R}} f(t)\, F(dt, A)\,. \tag{11}$$

for bounded Borel functions $f : \mathbf{R} \to \mathbf{C}$. Then $\tilde{I}_1(f, \cdot)$ is a complex measure. In fact $\tilde{I}_1 : \mathscr{B} \to (B(\mathbf{R}, \mathscr{B}, \mathbf{C}))^*$, the $B(\mathbf{R}, \mathscr{B}, \mathbf{C})$ being the Banach space of bounded complex Borel functions under the uniform norm, is a vector measure. So one can use the D-S integral (recalled at the beginning of the next section), defining

$$I_1(f, g) = \Big(\int_{\mathbf{R}} \bar{g}(t)\tilde{I}_1(dt)\Big)(f) \in \mathbf{C}\,, \qquad f, g \in B(\mathbf{R}, \mathscr{B}, \mathbf{C})\,. \tag{12}$$

Similarly starting with $F(A, \cdot)$ one can define $I_2(f, g)$. In general

$$I_1(f, g) \neq I_2(f, g)\,. \tag{13}$$

In fact the Fubini theorem does not hold in this context. For a counterexample, see ([27], §8). If there is equality in (13), then the pair (f, g) is said to be *integrable* relative to the bimeasure F, and the common value is denoted $I(f, g)$ and symbolically written as (f, g need not be bounded):

$$I(f, g) = \int_{\mathbf{R}}\int_{\mathbf{R}} f(s)\overline{g(t)}F(ds, dt)\,. \tag{14}$$

This is a *Morse-Transue* (or *MT-*) *integral.* While a characterization of MT-integrable functions is not easy, a good sufficient condition for this can be given as follows, (cf. [27], Thm. 7.1; [42], Théorème in §5.17). If f, g are step functions, so that $f = \sum_{i=1}^{n} a_i\chi_{A_i}$, $g = \sum_{j=1}^{n} b_j\chi_{B_j}$, then clearly $I(f, g)$ always exists and

$$I(f, g) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i\bar{b}_j\, F(A_i, B_j)\,. \tag{15}$$

Next define for any $\varphi \geqslant 0$, $\psi \geqslant 0$, Borel functions,

$$\tilde{I}(\varphi, \psi) = \sup\{|\, I(f, g)\,| : |\, f\,| \leqslant \varphi, |\, g\,| \leqslant \psi, f, g \text{ Borel step functions}\}\,,$$

and if u, v are any positive functions,

$$I^*(u, v) = \inf\{\tilde{I}(\varphi, \psi) : \varphi \geqslant u, \psi \geqslant v, \varphi, \psi \text{ are Borel}\}\,. \tag{16}$$

Now the desired result from the above papers is this: If (f, g) is a pair of complex Borel functions such that $I_1(f, g)$ and $I_2(f, g)$ exist in the sense of (12) and (13), and $I^*(|f|, |g|) < \infty$, then (f, g) is MT-integrable for the $\mathbf{C}$-bimeasure F. In the case that the bimeasure F is also positive definite and has *finite* semivariation, then each pair (f, g) of *bounded* complex Borel functions is MT-integrable relative to F. Moreover, using the notations of (7), one has

$$|\, I(f, g)\,| \leqslant \|\, F\,\| \cdot \|\, f\,\|_u \cdot \|\, g\,\|_u\,, \tag{17}$$

where $\| F \| = \| F \| (\mathbf{R} \times \mathbf{R})$. It should be noted, however, that the integrability of (f, g) generally need not imply that of $(|f|, |g|)$, and the MT-integral is *not* an absolutely continuous functional in contrast to the Lebesgue-Stieltjes theory, as already shown by counterexamples in [26] and [27]. Fortunately a certain dominated convergence theorem ([27], Thm. 3.3) is valid and this implies some density properties which can and will be utilized in our treatment below. Also f is termed F-integrable if (f, f) is MT-integrable. Our definition above is somewhat more restrictive than that of [27], but it suffices for this work. For the theory of [27], the space $B(\mathbf{R}, \mathscr{B}, \mathbf{C})$ in (12) and (13) is replaced by $C_{00}(\mathbf{R})$, its subset of continuous functions with compact supports, with the locally convex (inductive limit) topology. Note that, thus far, no special properties of $\mathbf{R}$ were used in the definition of the MT- integral, and the *definition and properties are valid if* $\mathbf{R}$ *is replaced by an arbitrary locally compact space (group in the present context).* This remark will be utilized later on.

With this necessary detour, the second concept can be given as follows:

Definition 2.2. A process $X : \mathbf{R} \to L_0^2(P)$, with $r(\cdot, \cdot)$ as its covariance function, is called *weakly harmonizable* if

$$r(s, t) = I(e^{is(\cdot)}, e^{it(\cdot)}) = \int_{\mathbf{R}} \int_{\mathbf{R}} e^{is\lambda - it\lambda'} F(d\lambda, d\lambda'), \; s, t \in \mathbf{R}, \tag{18}$$

relative to some positive definite bimeasure F of finite semivariation where the right side is the MT-integral.

In particular r is bounded and continuous (by (17) and Thm. 3.2 below). Moreover, if F is of bounded variation, then the MT-integral reduces to the Lebesgue-Stieltjes integral and (18) goes over to (3). The following work shows that the process of the counterexample following Definition 2.1 is weakly harmonizable. The same counterexample also shows that harmonizable processes generally do *not* admit shift operators on them, in that there need not be a continuous linear operator

$$\tau_s : X(t) \mapsto X(t+s) \in L_0^2(P), \qquad t \in \mathbf{R}$$

on $L_0^2(P)$. This is in distinction to certain other nonstationary processes of Karhunen type (cf. [9]).

3. Integral representation of a class of second order processes

In order to introduce and utilize the "V-boundedness" concept of Bochner's, it will be useful to have an integral representation of weakly harmonizable processes. This is done by presenting a comprehensive result for a more general