9. Multivariate extension and related problems

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9. MULTIVARIATE EXTENSION AND RELATED PROBLEMS

Here a multidimensional extension of weakly harmonizable processes and the filtering problem on them will be briefly discussed. Even though some results have direct k-dimensional analogs ($k \geqslant 2$), there are some new and non-trivial problems in this case for a successful application of the theory. The infinite dimensional case will not be considered here since the key finite dimensional problems are not well-understood and resolved.

Let $L_0^2(P, \mathbf{C}^k)$ $(= L_0^2(\Omega, \Sigma, P; \mathbf{C}^k))$ be the space of equivalence classes of measurable functions $f : \Omega \to \mathbf{C}^k$, the complex k-space, such that (i) $|f|^2 = \sum_{i=1}^{k} |f_i|^2$ is P-integrable, and (ii) $E(f) = \int_\Omega f(\omega)P(d\omega) = 0$, or equivalently,

$$E(f_i) = \int_\Omega f_i(\omega)P(d\omega) = 0, \qquad i = 1, ..., k\,,$$

where $f = (f_1, ..., f_k)$, $|f|$ is the Euclidean norm of f in $\mathbf{C}^k$, and (Ω, Σ, P) is a probability space. If $f, g \in L_0^2(P, \mathbf{C}^k)$, define $\|f\|_2^2 = (f, f)$ where the inner product is given by

$$(f, g) = \int_\Omega (f(\omega), g(\omega))P(d\omega) = \sum_{i=1}^{k} \int_\Omega f_i(\omega)\bar{g}_i(\omega)P(d\omega)\,.$$

Then $\mathscr{X} = L_0^2(P, \mathbf{C}^k)$ becomes a Hilbert space of k-vectors with zero means. If $k = 1$, one has the space considered in the preceding sections $(\mathscr{H} = L_0^2(P, \mathbf{C}))$.

Definition 9.1. Let G be an LCA group. Then a mapping $X : G \to \mathscr{X}$ is a *weakly* or *strongly harmonizable vector* (or k-dimensional) *random field* (or process) if for each $a = (a_1, ..., a_k) \in \mathbf{C}^k$, the mapping

$$Y_a = a \cdot X\left(= \sum_{i=1}^{k} a_i X_i\right) : G \to \mathscr{H}$$

is a (scalar) weakly or strongly harmonizable random field (or process).

Similarly a vector stationary, Karhunen, or class (C), processes are defined by reducing to the scalar cases.

It is immediate from this definition that the component processes are also harmonizably or stationarily etc. correlated according to the class they belong. Thus if r_a is the covariance function of the Y_a-process and R is the covariance matrix of the X-process, so that $r_a(g, h) = E(Y_a(g)\overline{Y_a(h)})$ and $R(g, h) = E(X^t(g)\bar{X}(h))$ where $X(g)$ is a k-th order (row) vector and "t" denotes the usual transpose of a vector or matrix, then $r_a(g, h) = aR(g, h)a^t$. With this notation, the integral representations of multivariate weakly and strongly harmonizable random fields can be obtained, using Theorem 3.3, in a straightforward manner.

THEOREM 9.2. *Let G be an LCA group and $X : G \to \mathscr{X} = L_0^2(P, \mathbf{C}^k)$, a weakly continuous bounded mapping. Then X is weakly harmonizable iff there is a stochastic measure $\tilde{Z}$ on $\hat{G} \to \mathscr{X}$ (or if $\tilde{Z}(A) = (Z_1(A), \ldots, Z_k(A)), A \subset \hat{G}$ is a Borel set, then each Z_j is a stochastic measure on $\hat{G} \to \mathscr{H}, j = 1, \ldots, k$), such that*

$$X(g) = \int_{\hat{G}} \langle g, s\rangle \tilde{Z}(ds), \qquad g \in G, \tag{99}$$

where $\hat{G}$ is the dual group of G. The mapping X is strongly harmonizable if further the matrix $F = (F_{jl}, j, l = 1, \ldots, k)$ with

$$F(A, B) = \big((Z_j(A), Z_l(B)), j, l = 1, \ldots, k\big)$$

is of bounded variation on $\hat{G}$, or equivalently each F_{jl} is of bounded variation on $\hat{G}$. The covariance matrix R is representable as:

$$R(g, h) = \int_{\hat{G}} \int_{\hat{G}} \langle g, s\rangle \overline{\langle h, t\rangle}\, F(ds, dt), \qquad g, h \in G, \tag{100}$$

where the right side is the MT-integral, or the Lebesgue-Stieltjes integral, defined componentwise, accordingly as X is weakly or strongly harmonizable, and where F is a positive definite matrix of bounded bimeasures or of Lebesgue-Stieltjes measures. Conversely, if $R(\cdot, \cdot)$ is a positive definite matrix representable as (100), *then it is the covariance matrix of a multivariate harmonizable random field.*

Sketch of proof: Let $a \in \mathbf{C}^k$ be arbitrarily fixed and consider

$$Y_a = a \cdot X (= aX^t).$$

If X is weakly harmonizable, so that Y_a is also, then by Theorem 3.3 (trivially extended when $\mathbf{R}$ is replaced by G), there is a stochastic measure Z_a on $\hat{G} \to \mathscr{H}$ such that

$$Y_a(g) = \int_{\hat{G}} \langle g, s\rangle Z_a(ds), \qquad g \in G.$$

From this and the definition of Y_a, it follows that $Z_{(\cdot)}(A) : \mathbf{C}^k \to \mathscr{H}$ is linear and continuous. Hence there is a $\tilde{Z}$ on $\hat{G} \to \mathscr{X}^{**}$ ($= \mathscr{X}$, by reflexivity) such that $Z_a(A) = a \cdot \tilde{Z}(A)$, and it is evident that $\tilde{Z}$ is σ-additive on $\mathscr{B}(\hat{G}) \to \mathscr{X}$ so that it is a stochastic measure. It follows from the properties of the D-S integral that:

$$Y_a(g) = a \cdot X(g) = \int_{\hat{G}} \langle g, s\rangle a \cdot \tilde{Z}(ds) = a \cdot \int_{\hat{G}} \langle g, s\rangle Z(ds), \tag{101}$$

where the last integral defines an element of $\mathscr{X}$. This implies (99) since "a" is arbitrary and $X(\cdot)$ as well as the integral operator are continuous. The converse is similarly deduced from the corresponding part of Theorem 3.3.

If X is strongly harmonizable, then so is Y_a and if F_a is its covariance bimeasure, then $F_a = aF\bar{a}^t$ where

$$F(A, B) = \left((Z_j(A), Z_l(B)), j, l = 1, 2, \ldots, k\right).$$

Now taking special values for a in $\mathbf{C}^k$, it follows immediately that each component F_{jl} of F is of bounded variation. Interpreting (100) componentwise, the result follows from the scalar case. The same representation holds with the MT-integration for the weakly harmonizable case. All other statements, including the converses, are similarly deduced. This terminates the sketch.

By an analogous reasoning, it is evidently possible to assert that there is a 2-majorant of $\tilde{Z}$, and the X-process has a (vector) stationary dilation. These results are of real interest in the context of the important filtering problem which can be abstractly stated following Bochner [2].

If $X : G \to \mathscr{X}$ is a random field, a (not necessarily bounded) linear operator $\Lambda : \mathscr{X} \to \mathscr{X}$ is called a *filter* of X, if Λ commutes with the translation operator on X, i.e., if $(\tau_h X)(g) = X(hg)$, then $\tau_h(\Lambda X) = \Lambda(\tau_h X)$, where the domain

$$\operatorname{dom}(\Lambda) \supset \{\tau_h X(g), g \in G, h \in G\}.$$

The problem is to find solutions X of the equation:

$$\Lambda X = Y(\in \mathscr{X}), \tag{102}$$

such that if Y is a given weakly or strongly harmonizable random field so must X be.

For the stationary case, a general concept of filter was discussed by Hannan [11]. If $k = 1$, $\Lambda = \sum_{i=1}^{m} a_i \Delta_i$ is a reverse shift operator with $G = \mathbf{R}$ (so $\Delta_i X(t) = X(t-i)$) and Y is stationary, then this problem was completely solved by Nagabhushanam [28], and by Kelsh [19] in the strongly harmonizable case. In both these studies, the conditions are on the measure function F of (33). If $k \geqslant 2$, under the usual assumptions on the random fields, the following new questions arise with (99) and (100). Frequently employed general forms of Λ include the constant coefficient difference, differential, or integral operators, or a mixture of these. For instance, if $\Lambda = \sum_{j=0}^{m} A_j D^j$, where the A_j are k-by-k constant matrices, and $D^j = \dfrac{d^j}{dt^j}$, $(G = \mathbf{R})$ then (102) takes the following form in order that it admit a (weakly) harmonizable solution for a harmonizable Y where $X^{(j)}$ denotes the mean-square j-th derivative (assumed to exist):

$$\int_{\mathbf{R}} e^{it\lambda} Z_y(d\lambda) = Y(t) = (\Lambda X)(t) = \sum_{j=0}^{m} A_j X^{(j)}(k-j)$$
$$= \sum_{j=0}^{m} A_j \int_{\mathbf{R}} e^{i(t-j)\lambda}(i\lambda)^j Z_x(d\lambda)$$
$$= \int_{\mathbf{R}} T(\lambda) \cdot e^{it\lambda} Z_x(d\lambda)\,, \tag{103}$$

where $T(\lambda) = \sum_{j=0}^{m} A_j e^{-ij\lambda}(i\lambda)^j$, called the *generator* of Λ in [2], and Z_x, Z_y are the representing stochastic measures of X- and Y-processes. Clearly the existence of solutions of (102) depends on the coefficients A_j's determining the analytical properties of the generator $T(\cdot)$. If the process is strongly harmonizable then (103) implies (*-denoting conjugate transpose)

$$R_y(s, t) = \int_{\mathbf{R}} \int_{\mathbf{R}} e^{is\lambda - it\lambda'} F_y(d\lambda, d\lambda')$$
$$= \int_{\mathbf{R}} \int_{\mathbf{R}} e^{is\lambda} T(\lambda) F_x(d\lambda, d\lambda') \left(T(\lambda')e^{it\lambda'}\right)^*\,, \tag{104}$$

where F_x and F_y are the k-by-k matrix covariance bimeasures of X- and Y-processes. For a special class of strongly harmonizable k-vector processes, recently Kelsh [19] found sufficient conditions on the generator $T(\cdot)$ for a solution of (102) when differential operators are replaced by difference operators so that $\{\lambda : T(\lambda) = 0\}$ is finite. The solution here hinges on the properties of the structure of the space:

$$L^2(F_x) = \{T : \mathbf{R} \to B(\mathbf{C}^k)\,,\ \| \int_{\mathbf{R}} \int_{\mathbf{R}} T(\lambda) F_x(d\lambda, d\lambda') T^*(\lambda') \| < \infty\}\,. \tag{105}$$

Since the integral in (105) defines a positive (semi-) definite matrix, its trace gives a semi-norm. The measure function F being a matrix bimeasure, several new problems arise for an analysis of the $L^2(F_x)$-space. For the weakly harmonizable case, an extension of the MT-integration, to include such integrals, should be established. The resulting theory can then be utilized for the multivariate filtering study. Even if $k = 2$, the problem is non-trivial, involving the rank questions of F_x. Application of the dilation results to the filtering problem has some novel features, but it does not materially simplify the problem.

Another interesting point is to seek "weak solutions" of the filtering equation (102) in the sense of distribution theory. This idea is introduced in [2]. If $\mathcal{G}$ is a class of functions on $\mathbf{R}$ (e.g. the Schwartz space $C_{00}^{\infty}(\mathbf{R})$) with a locally convex topology, then one says that (102) has a (weak) solution iff for each $f \in \mathcal{G}$

$$\int_{\mathbf{R}} f(t) Y(t) dt = \int_{\mathbf{R}} f(t) \Lambda X(t) dt = \int_{\mathbf{R}} (\bar{\Lambda} f)(t) X(t) dt\,, \tag{106}$$

where $\tilde{\Lambda} : \mathscr{G} \to \mathscr{G}$ is an operator, associated with Λ, defined by the last two integrals above. It is an "adjoint" to Λ. For instance, if Λ is a differential operator with $T(\cdot)$ as its generator, if $k = 1$ and X, Y are stationary, then $\tilde{\Lambda}$ is given by

$$(\tilde{\Lambda} f)(t) = \int_{\mathbf{R}} T(t-\lambda) f(\lambda) F_x(d\lambda), \qquad f \in \mathscr{G} \tag{107}$$

where F_x is the spectral measure function of the X-process. Clearly many other possibilities are available. Thus there are a number of directions to pursue the research on these problems, and the paper [2] has a wealth of ideas of great interest here.

This essentially includes what is known about weakly harmonizable random fields and processes, as far as their structure is concerned. Since the class (C) of Cramér and its weak counterpart (cf. Definition 3.1) and the Karhunen class of processes, defined by (31), are natural generalizations of harmonizable and stationary classes, it is reasonable to ask whether the latter is a dilation of the former, i.e., is the analog of Theorem 6.1 true for weakly class (C)? A restricted version can be establshed by the same methods, but the parallel generalization does not hold. (See [38] on this point.) This question will be briefly discussed here in order to include it in the set of problems raised by the present study.

Recall that a mapping $X : \mathbf{R} \to L_0^2(P)$ is a *Karhunen process* if its covariance function $r(\cdot, \cdot)$ admits a representation

$$r(s, t) = \int_{\mathbf{R}} g_s(\lambda) \overline{g_t(\lambda)} F(d\lambda), \qquad s, t \in \mathbf{R},$$

relative to a family $\{g_s(\cdot), s \in \mathbf{R}\}$ of measurable functions and F which defines a locally finite positive regular (or Radon) measure on $\mathbf{R}$ and $g_s \in L^2(F)$ (cf. also [10], p. 241). As an immediate consequence of Theorem 3.2 (cf. Remark 2 following its proof), an integral representation for Karhunen processes can be given.

PROPOSITION 9.3. *Let S be a locally compact space and $X : S \to L_0^2(P)$ be a process of Karhunen class relative to a locally finite positive regular (or Radon) measure F on S and a family $\{g_t, t \in S\} \subset L^2(F)$, the space of all scalar square integrable functions on $(S, \mathscr{B}, F)$. Then there is a locally bounded regular (or Radon) stochastic measure $Z : \mathscr{B}_0 \to L_0^2(P)$ where $\mathscr{B}_0 \subset \mathscr{B}$ is the δ-ring of bounded sets, such that* (i)

$$E(Z(A) \cdot \bar{Z}(B)) = F(A \cap B), \qquad A, B \in \mathscr{B}_0,$$

i.e., Z is orthogonally scattered, and (ii) *one has*

$$X(t) = \int_S g_t(\lambda) Z(d\lambda), \qquad t \in S, \tag{108}$$

where the right side symbol is a D-S *integral (cf. also* [42], §1*). Conversely, if* $X : S \to L_0^2(P)$ *is a process defined by* (108) *relative to an orthogonally scattered measure* Z *on* S *and* $\{g_t, t \in S\}$ *satisfies the above conditions, then it is a Karhunen process with respect to the family* $\{g_t, t \in S\}$ *and* F *defined by* $F(A \cap B) = (Z(A), Z(B))$. *Moreover*

$$\mathscr{H}_X = \overline{sp}\{X(t), t \in S\} \subseteq \mathscr{H}_Z = \overline{sp}\{Z(A), A \in \mathscr{B}_0\} \subset L_0^2(P)$$

and $\mathscr{H}_X = \mathscr{H}_Z$ *iff* $\{g_t, t \in S\}$ *is dense in* $L^2(F)$.

A proof of this result is essentially given in ([10], p. 242) and is a simplification of that of Theorem 3.2. Even a multidimensional version is not difficult, which in fact is analogous to that of Theorem 9.2 above. Actually, the version in [10] is sketched for the k-dimensional case.

It follows from the arguments of the D-S theory of integration that a bounded linear operator T and the vector integral such as that of (108) commute even if Z is of locally finite semivariation on the locally compact space S. This extension of ([8], IV.10) was proved in ([42], p. 79), and shown to be easy. Thus if $X : S \to L_0^2(P)$ is a Karhunen process, so that it is representable as in (108) and if $T \in B(L_0^2(P))$, then it follows that

$$TX(t) = \int_S g_t(\lambda) T \circ Z(d\lambda), \tag{109}$$

and it is simple to see that $\tilde{Z} = T \circ Z$ is a stochastic measure of locally finite semivariation, but not necessarily orthogonally scattered. Hence by Theorem 3.2, TX is weakly of class (C).

In the opposite direction, for a process $\{X(s), s \in S\} \in$ weakly class (C), one cannot apply the theory of Section 5 above if only $\{g_t, t \in S\} \subset L^2(F_x)$, and no further restrictions are imposed, where $L^2(F_x)$ is the space of functions g such that $|g|$ is MT-integrable relative to the covariance bimeasure F_x representing X (cf. (105), with $k = 1$). Suppose now that F_x is such that if each g_t is a *bounded* Borel function and $M(S)$ is the uniformly closed algebra generated by $\{g_t, t \in S\}$ then $M(S) \subset L^2(F_x)$. Let

$$Tg_t = X(t) = \int_S g_t(\lambda) Z(d\lambda)$$

and extend T linearly to $M(S)$. Then $T \in B(M(S), \mathscr{H})$ when $M(S)$ is given the uniform norm. This forces F_x to be of finite semivariation if at least one g_s has noncompact support. Under this assumption T is a 2-absolutely summing, and Proposition 5.6 is applicable. Hence

$$\| Tf \| \leqslant \| f \|_{2,\mu}, \qquad f \in M(S) \tag{110}$$

for a finite measure μ on S. (A similar result seems possible *if* Z is restricted so that $T \in B(L^2(F_x), \mathscr{H})$, defined above is Hilbert-Schmidt by [22], p. 302. But it is not a good assumption here.) Thus one can repeat the proof of Theorem 6.1 essentially verbatim and establish a dilation result. Omitting the details of this computation one obtains the following result. (For related remarks, details and other results, see [38].)

THEOREM 9.4. *Let* S *be a locally compact space and*

$$X : S \to L_0^2(P) = \mathscr{H}$$

be a Karhunen process relative to a Radon measure F *and a family*

$$\{g_t, t \in S\} \subset L^2(F).$$

If $Q : \mathscr{H} \to \mathscr{H}$ *is any (bounded) projection, then* $\tilde{X}(t) = QX(t), t \in S,$ *is a process in weakly class (C). On the other hand if* $\{X(t), t \in S\}$ *is an element of weakly class (C), and so is representable in the form (108) for some family* $\{g_t, t \in S\} \subset L^2(F_x)$ *where* F_x *is a bounded covariance bimeasure of the process* ($L^2(F_x)$ *is defined above), and if each* g_t *is also bounded, then there exists an extension Hilbert space* $\mathscr{K} \supset \mathscr{H}$, *a probability space* $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{P})$ *with* $\mathscr{K} = L_0^2(\tilde{P})$, *and a Karhunen process* $Y : S \to \mathscr{K}$ *such that*

$$X(t) = QY(t), \qquad t \in S,$$

where Q *is the orthogonal projection on* $\mathscr{K}$ *with range* $\mathscr{H}$.

This result points out clearly the need to consider the domination problem for other Banach spaces than those covered by the results of Section 5. Indeed the associated abstract problem of classifying Banach spaces admitting a positive p-majorizable measure for each vector measure from a probability space into that space is essentially open. Also the preceding theorem and related analysis presumably extend to class$_N$ (C)-processes of Definition 3.4. This will be of independent interest in addition to its use in a treatment of the general filtering theory on these processes. Other problems noted in the main text of the paper are of both methodological and applicational importance for a future study.