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ON THE NUMBER OF RESTRICTED PRIME FACTORS OF AN INTEGER. III

by Karl K. Norton

§1. Introduction

Let P be the set of all (positive rational) prime numbers, and let E be an arbitrary nonempty subset of P. Throughout this paper, let p denote a general member of P, and for non-negative integers a, write $p^a \parallel n$ if $p^a \mid n$ and $p^{a+1} \nmid n$. For each positive integer n, define

$$\omega(n; E) = \sum_{p \mid n,\ p \in E} 1, \qquad \Omega(n; E) = \sum_{p^a \parallel n,\ p \in E} a .$$

We usually write $\omega(n; P) = \omega(n), \Omega(n; P) = \Omega(n)$. In this paper, we shall estimate the functions

$$\begin{aligned} S(x, y; E, \omega) &= \operatorname{card}\{n \leqslant x : \omega(n; E) > y\}, \\ S(x, y; E, \Omega) &= \operatorname{card}\{n \leqslant x : \Omega(n; E) > y\} \end{aligned} \tag{1.1}$$

when y is appreciably larger than the normal order of $\omega(n; E)$ and $\Omega(n; E)$; y may even be as large as the maximum order of $\omega(n; E)$ or $\Omega(n; E)$, respectively. (Here and throughout, card B means the number of members of the set B, and if $Q(n)$ is a statement about the integer n, we often write $\{n \leqslant x : Q(n)\}$ instead of $\{n : 1 \leqslant n \leqslant x \text{ and } Q(n)\}$.)

Define

$$E(x) = \sum_{p \leqslant x,\ p \in E} p^{-1} \qquad (x \text{ real}) . \tag{1.2}$$

In [13], it was observed that if $E(x) \to +\infty$ as $x \to +\infty$, then both the average order and the normal order of $\omega(n; E)$ are equal to $E(n)$, and the same statement holds for $\Omega(n; E)$. In [13], we obtained sharp inequalities for the functions (1.1) when $0 < y < 2E(x)$, roughly. In [14], we gave asymptotic formulas for the same functions when $E(x) \to +\infty$ and $y = E(x) + o(E(x))$ as $x \to +\infty$. It is well-known, however, that

$$E(x) \leqslant \log\log x + O(1) \qquad \text{for} \quad x \geqslant 2 ,$$

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whereas if x is large, $\omega(n; E)$ and $\Omega(n; E)$ may be much larger than log log x for some values of $n \leqslant x$. For example, the method of [6, pp. 262-263, 359] shows that

$$\limsup_{n \to +\infty} \frac{\omega(n) \log \log n}{\log n} = 1, \tag{1.3}$$

and a more precise version of (1.3) was obtained in [12, pp. 96-100]. (See also the remarks at the beginning of §3 below.) Before stating estimates for the functions (1.1) when y is large, it seems worthwhile to generalize results like (1.3) to $\omega(n; E)$. First define

$$\pi(x; E) = \sum_{p \leqslant x,\, p \in E} 1 \qquad (x \text{ real}), \tag{1.4}$$

and write

$$\log_2 x = \log \log x, \qquad \log_r x = \log(\log_{r-1} x) \quad \text{for} \quad r = 3, 4, \ldots . \tag{1.5}$$

THEOREM 1.6. *Suppose that there exists a real number* $\gamma(E) > 0$ *such that*

$$\pi(x; E) = \gamma(E)(x/\log x)\{1 + O_E(1/\log x)\} \quad \text{for all} \quad x \geqslant 2. \tag{1.7}$$

Then for each $n \geqslant 3$, *we have*

$$\omega(n; E) \leqslant \frac{\log n}{\log_2 n} + \frac{\{1 + \log \gamma(E)\} \log n}{(\log_2 n)^2} + O_E\left(\frac{\log n}{(\log_2 n)^3}\right), \tag{1.8}$$

with equality for infinitely many n.

Here and throughout, the notation $O_{\delta, \varepsilon, \ldots}$ implies a constant depending at most on $\delta, \varepsilon, \ldots$, while O without subscripts implies an *absolute* constant. Likewise, for $i = 1, 2, \ldots$, we shall write $c_i(\delta, \varepsilon, \ldots)$ for a positive number depending at most on $\delta, \varepsilon, \ldots$, while c_i will mean a positive absolute constant.

It is interesting to observe that a much weaker hypothesis than (1.7) still implies that the maximum order of $\omega(n; E)$ is approximately $(\log n)(\log_2 n)^{-1}$. See the remarks after the proof of Theorem 1.6 in §3.

After (1.3) and Theorem 1.6, it is natural to ask how often $\omega(n; E)$ and $\Omega(n; E)$ assume values appreciably larger than their normal order $E(n)$. It appears that rather little was known about this problem until very recently. The earliest contribution was by Hardy and Ramanujan [5] (reprinted in [15, pp. 262-275]), whose estimate for card $\{n \leqslant x : \omega(n) = m\}$ leads easily to a good upper bound for $S(x, y; P, \omega)$ (essentially the same as the bound given in Theorem 1.14 below). However, they did not state explicitly a result of the latter type. For arbitrary E, much weaker upper bounds for $S(x, y; E, \omega)$ and $S(x, y; E, \Omega)$ can be derived from a general theorem of Turán [19] on the distribution of values of additive functions. (See also Turán [18] or Hardy and Wright [6, pp. 356-358] for the case $E = P$, and see [13, §§1, 3] and [14, pp. 18-19] for remarks on all of this early work.) For the particular functions $\omega(n; E)$ and $\Omega(n; E)$, Turán's bounds were improved considerably in the author's paper [13; (5.16), (5.15), (1.11)], where it was observed that for any set E,

$$S(x, \alpha E(x); E, \omega) \leqslant x \exp\{(\alpha - 1 - \alpha \log \alpha) E(x)\} \tag{1.9}$$

for real $x \geqslant 1$, $\alpha \geqslant 1$, where $E(x)$ is defined by (1.2). A similar (slightly less precise) result was stated for $\Omega(n; E)$ when $1 \leqslant \alpha < p_1$, where p_1 is the smallest member of E. No lower bound was obtained in either case for $\alpha \geqslant 2$, so that the precision of (1.9) for large α was not clear. In a later paper [2], Erdös and Nicolas obtained a rather good estimate in the special case $E = P$. They showed that for any fixed α with $0 < \alpha < 1$,

$$\text{card}\{n \leqslant x : \omega(n) > \alpha (\log x)(\log_2 x)^{-1}\} = x^{1-\alpha+o(1)} \tag{1.10}$$

as $x \to +\infty$. (In fact, they obtained a somewhat more precise result resembling Theorem 4.13 below.) However, they did not get an analogous result for $\Omega(n)$, nor did they generalize to $\omega(n; E)$ or $\Omega(n; E)$. Furthermore, their method did not give good upper estimates for $S(x, y; P, \omega)$ when y is appreciably smaller than $(\log x)(\log_2 x)^{-1}$. We propose to remedy all of these drawbacks to some extent. First, we obtain the following lower bound by a refinement of the Erdös-Nicolas method:

THEOREM 1.11. *Suppose that there exists a real number* $\gamma(E) > 0$ *such that* (1.7) *holds. Let* $\varepsilon > 0$, *and suppose that* $x \geqslant c_1(E, \varepsilon)$ *and*

$$\begin{aligned} c_2(E) \leqslant y \leqslant{}& (\log x)(\log_2 x)^{-1} \\ &+ \{1 + \log \gamma(E) - \varepsilon\} (\log x)(\log_2 x)^{-2}. \end{aligned} \tag{1.12}$$

Then

$$\begin{aligned} S(x, y; E, \omega) \geqslant x \exp\{&-y(\log y + \log_2 y - \log \gamma(E) - 1) \\ &+ O_E(y(\log_2 y)/\log y)\}. \end{aligned} \tag{1.13}$$

(1.8) shows that only a very small weakening of the hypothesis (1.12) would be of any interest. In Theorem 3.20, we assume much less than (1.7) and derive a result similar to Theorem 1.11 (but somewhat weaker).

Concerning upper bounds for $S(x, y; E, \omega)$, we have obtained only a modest improvement of (1.9); see Theorem 4.8 and Corollary 4.12. It should be emphasized that (1.9) and Theorem 4.8 hold for an arbitrary set E (without the assumption (1.7)). Using the same methods, we deduce

THEOREM 1.14. *Suppose that there exists a real number* $\gamma(E) > 0$ *such that* (1.7) *holds. If* $x \geqslant 3$ *and* $y \geqslant \gamma(E) \log_2 x$, *then*

$$S(x, y; E, \omega) \leqslant x \exp\{-y(\log y - \log_3 x - \log \gamma(E) - 1) - \gamma(E) \log_2 x + O_E(y/\log_2 x)\}. \tag{1.15}$$

Although there is a considerable gap between (1.13) and (1.15), the results are more general and somewhat sharper than those of Erdös and Nicolas [2]. In particular, we get a generalization of (1.10) (see Theorem 4.13). Theorems 1.11 and 1.14 also yield immediately the following result which could not be obtained by the Erdös-Nicolas method:

COROLLARY 1.16. *Suppose that there exists a real number* $\gamma(E) > 0$ *such that* (1.7) *holds. If* $0 < \alpha < 1$ *and* $x \geqslant c_3(E, \alpha)$, *then*

$$S(x, (\log x)^\alpha; E, \omega) = x \exp\{-\alpha (\log x)^\alpha \log_2 x + O((\log x)^\alpha \log_3 x)\}.$$

It should be mentioned that when $E = P$ (the set of all primes) and $y/\log_2 x$ is bounded and not too close to 1, Theorems 1.11 and 1.14 can be replaced by a striking asymptotic formula which was recently obtained by H. Delange (for the proof, see [2]):

THEOREM 1.17 (Delange). *Let* x, α, r_1, r_2 *be real with* $x \geqslant 3$, $1 < r_1 \leqslant \alpha \leqslant r_2$. *Then*

$$S(x, \alpha \log_2 x; P, \omega) = \frac{F(\alpha)\, \alpha^{1/2 + \alpha \log_2 x - [\alpha \log_2 x]}}{(2\pi)^{1/2} (\alpha - 1)} \cdot \frac{x}{(\log x)^{1 - \alpha + \alpha \log \alpha} (\log_2 x)^{1/2}} \left\{1 + O_{r_1, r_2}\left(\frac{1}{\log_2 x}\right)\right\},$$

where $[z]$ *means the largest integer* $\leqslant z$ *and*

$$F(\alpha) = \frac{1}{\Gamma(\alpha+1)} \prod_p \left(1 + \frac{\alpha}{p-1}\right)\left(1 - \frac{1}{p}\right)^{\alpha}.$$

Delange obtained a similar result for card $\{n \leqslant x : \omega(n) \leqslant \alpha \log_2 x\}$ when $x \geqslant 3$, $(\log_2 x)^{-1} \leqslant \alpha \leqslant r_3 < 1$ (see [2]). In this connection, it is interesting to note the estimate

$$F(\alpha) = \exp\{-\alpha \log \alpha - \alpha \log_2 \alpha + (1-\gamma)\alpha + O(\alpha/\log \alpha)\}$$

for real $\alpha \geqslant 2$, where γ is Euler's constant. (Some effort is required to show this, and we omit the proof.)

For values of α near 1, Kubilius [8, Theorem 9.2] proved a result on the distribution of $\omega(n)$ which is similar to Theorem 1.17. His theorem was later extended by himself [9] and Laurinčikas [10] to somewhat more general additive functions, and it was generalized to $\omega(n; E)$ and $\Omega(n; E)$ by Norton [14]. The estimates for $S(x, y; E, \omega)$ derived in the present paper are not as precise as Theorem 1.17 or the earlier work cited, but they are more general with respect to E (except for [14]), and they hold for much larger values of y.

We now consider the function $\Omega(n; E)$. Here we assume that E is any nonempty set of primes; in particular, nothing like (1.7) is assumed. For completeness, we begin by stating the following easy result:

THEOREM 1.18. *Let* p_1 *be the smallest member of* E. *Then*

$$\Omega(n; E) \leqslant (\log n)(\log p_1)^{-1} \qquad \text{for all} \qquad n \geqslant 1, \tag{1.19}$$

with equality if and only if $n = p_1^a$ *for some integer* $a \geqslant 0$.

This follows from

$$n \geqslant \prod_{p^a \| n,\, p \in E} p^a \geqslant \prod_{p^a \| n,\, p \in E} p_1^a = p_1^{\Omega(n; E)}.$$

We now proceed to estimate $S(x, y; E, \Omega)$ (defined by (1.1)). For $y \geqslant 2E(x)$, rather little previous work has been done on this problem, and all of it was restricted to the special case $E = P$ (the set of all primes). Selberg [17, p. 87] stated without detailed proof the following asymptotic formula:

$$\text{card}\{n \leqslant x : \Omega(n) = m\} \sim A 2^{-m} x \log x$$

for integers m satisfying $(2+\varepsilon)\log_2 x \leqslant m \leqslant B \log_2 x$. (Here $\varepsilon > 0$ is arbitrarily small, while A and B are positive absolute constants; it is not clear from [17] how large B could be.) Selberg also gave an asymptotic formula for card $\{n \leqslant x : \omega(n) = m\}$ when $m/\log_2 x$ is bounded. His work was recently extended to considerably larger values of m (roughly $m < (\log x)^{3/5}$) by Kolesnik and Straus [7], whose theorems are quite complicated. These results, together with the formula

$$S(x, y; P, \Omega) = \sum_{y < m \leqslant Y} \operatorname{card}\{n \leqslant x : \Omega(n) = m\} + S(x, Y; P, \Omega)$$

and different tools for estimating $S(x, Y; P, \Omega)$ from above, would yield some information about $S(x, y; P, \Omega)$. However, it appears that neither [17] nor [7] would thus lead to an estimate for $S(x, y; P, \Omega)$ which is both simple and reasonably precise when $y/\log_2 x$ is unbounded. To the best of our knowledge, the only previous result of the latter type is due to Erdös and Sárközy [3], who recently proved that

$$S(x, y; P, \Omega) \leqslant c_4 y^4 2^{-y} x \log x \quad \text{for} \quad x \geqslant 3, y \geqslant 1. \tag{1.20}$$

We shall generalize their work to $S(x, y; E, \Omega)$ and get a sharper upper bound. Although the result could be phrased in terms of the function $E(x)$ (defined by (1.2)), it is more convenient to state it in terms of a real number v which in practice is taken to be an approximation to $E(x)$. (For example, if $E = P$, we could take $v = \log_2 x$.)

THEOREM 1.21. *Let x, v, y be real with $x \geqslant 1, v \geqslant 1$, and $y \geqslant 0$. Let p_1 be the smallest member of E, and define*

$$\Lambda = \Lambda(x, v; E) = \max\{2, |E(x) - v|\}. \tag{1.22}$$

Then

$$S(x, y; E, \Omega) \leqslant c_5(p_1) p_1^{-y} x v^{1/2} e^{(p_1 - 1)v + p_1 \Lambda}. \tag{1.23}$$

We remark that (1.23) is our best upper bound when $y > p_1 v - v^{1/2}$, but it can be improved for smaller values of y (see Lemma 5.3).

Concerning the problem of estimating $S(x, y; E, \Omega)$ from below, we shall state only the following simple result:

THEOREM 1.24. *Let p_1 be the smallest member of E. If $x \geqslant p_1$ and $0 \leqslant y \leqslant (\log x)(\log p_1)^{-1} - 1$, then*

$$S(x, y; E, \Omega) \geqslant (1/2) p_1^{-y-1} x.$$

To prove this, let $k = [y] + 1$ (so k is the smallest integer greater than y), and observe that the multiples n of p_1^k have the property that $\Omega(n; E) \geqslant k > y$.

There are just $[xp_1^{-k}]$ of these $n \leqslant x$, and since $[z] \geqslant z/2$ for $z \geqslant 1$, we get the result.

It is clear that Theorem 1.24 is essentially best possible in certain extreme cases (for example, if $E = \{p_1\}$, or if $x = p_1^a$ and $y = a - 1$).

When $E = P$ (the set of all primes), we can take $v = \log_2 x$. Then $\Lambda = O(1)$, and we have the following corollary of Theorems 1.21 and 1.24:

COROLLARY 1.25. *If* $x \geqslant e^e$ *and* $0 \leqslant y \leqslant (\log x)(\log 2)^{-1} - 1$, *then*

$$2^{-y-2} x \leqslant S(x, y; P, \Omega) \leqslant c_6 2^{-y} x (\log x)(\log_2 x)^{1/2}.$$

Corollary 1.25 should be compared with the Erdös-Sárközy result (1.20) and with the asymptotic formula of Selberg mentioned after Theorem 1.18. When $y < 2 \log_2 x$ (roughly), more precise estimates for $S(x, y; P, \Omega)$ can be obtained from [13] and [14].

In a later paper, we shall show that if p_1 is the smallest member of E and $\varepsilon > 0$ is fixed, then the precise order of magnitude of $S(x, y; E, \Omega)$ is

$$p_1^{-y} x \exp\{(p_1 - 1) E(x)\}$$

when $E(x)$ is sufficiently large and

$$p_1 E(x) \leqslant y \leqslant (1 - \varepsilon)(\log x)(\log p_1)^{-1}.$$

This theorem is much more difficult to prove than Theorem 1.21. Its proof depends on Theorem 1.21 and on an extension of Halász's work [4] concerning the local distribution of $\Omega(n; E)$. Theorem 1.21 remains our best upper bound when y is close to $(\log x)(\log p_1)^{-1}$ (cf. Theorem 1.18), and it seems to be the most we can achieve by a fairly simple method.

§2. NOTATION

The symbols a, m, n always represent integers with $a \geqslant 0, m \geqslant 0, n > 0$. The letter p always denotes a prime, while $v, w, x, y, z, \alpha, \beta, \delta, \varepsilon, \sigma$ are real numbers. $[x]$ means the largest integer $\leqslant x$. The notation $\log_r x$ is defined by (1.5), and the notations $O, O_{\delta, \varepsilon, \ldots}, c_i, c_i(\delta, \varepsilon, \ldots)$ are explained after Theorem 1.6. If a condition such as "$x \geqslant c_i(\delta, \varepsilon, \ldots)$" is used as a hypothesis, it is to be understood that $c_i(\delta, \varepsilon, \ldots)$ is sufficiently large. We shall occasionally use the notations $\ll$, $\gg$ to imply constants which are *absolute*. (Thus $A = O(B)$ is equivalent to $A \ll B$.)

Empty sums mean 0, empty products 1, and we define $0^0 = 1$. The notation

$$x_1 \cdots x_m / y_1 \cdots y_n$$

is sometimes used instead of

$$(x_1 \cdots x_m)(y_1 \cdots y_n)^{-1}.$$

Throughout this paper, E denotes a nonempty set of primes, to be regarded as quite arbitrary unless further assumptions are stated. $E(x)$ is always defined by (1.2). p_1 always means the smallest member of E, and if

$$E - \{p_1\} = \{p : p \in E \quad \text{and} \quad p \neq p_1\}$$

is not empty, then p_2 denotes the smallest member of $E - \{p_1\}$. When x and v are positive, the function $\Lambda = \Lambda(x, v; E)$ is always defined by (1.22).

§3. Proofs of Theorems 1.6 and 1.11, and related results

Before proving (1.8), we observe that a similar but weaker inequality has a very simple proof. For if $y > 1$, then

$$\log n \geqslant \sum_{p|n} \log p \geqslant \sum_{p|n,\, p \geqslant y} \log p \geqslant (\log y) \sum_{p|n,\, p \geqslant y} 1,$$

and hence

$$\omega(n) = \sum_{p|n,\, p<y} 1 + \sum_{p|n,\, p \geqslant y} 1 \leqslant y + (\log n)(\log y)^{-1}.$$

The right-hand side is approximately minimized by taking

$$y = (\log n)(\log_2 n)^{-2},$$

and we obtain

$$\omega(n) \leqslant \frac{\log n}{\log_2 n}\left\{1 + O\left(\frac{\log_3 n}{\log_2 n}\right)\right\} \quad \text{for} \quad n \geqslant 16\,(>e^e). \tag{3.1}$$

Another simple proof of (3.1) can be based on Newman's observation [11, p. 652] that if $\omega(n) = r$, then $n \geqslant r!$.

To get the sharper inequality (1.8), it seems to be necessary to use an assumption such as (1.7) about the distribution of E. First we need a lemma relating $\pi(x; E)$ (defined by (1.4)) and

$$\theta(x; E) = \sum_{p \leqslant x,\, p \in E} \log p. \tag{3.2}$$

LEMMA 3.3. *Suppose that there exists a real number* $\gamma(E) > 0$ *such that* (1.7) *holds. Then for* $x > c_{11}(E)$,

$$\theta(x;E) = \pi(x;E)\left\{\log \pi(x;E) + \log_2 \pi(x;E) - 1 - \log \gamma(E) + \frac{\log_2 \pi(x;E)}{\log \pi(x;E)} + O_E\left(\frac{1}{\log \pi(x;E)}\right)\right\}. \tag{3.4}$$

Proof: For notational simplicity, we write $l_r = \log_r x$, $L_r = \log_r \pi(x;E)$ whenever these are defined. First note that for $x > c_{12}(E)$, (1.7) implies

$$L_1 = l_1 - l_2 + \log \gamma(E) + O_E(1/l_1). \tag{3.5}$$

In particular, $L_1 \sim l_1$ and $L_2 \sim l_2$ as $x \to +\infty$, so for $x > c_{13}(E)$, (3.5) implies

$$L_1 = l_1\{1 + O_E(L_2/L_1)\},$$

and multiplication by $(L_1 l_1)^{-1}$ yields

$$l_1^{-1} = L_1^{-1}\{1 + O_E(L_2/L_1)\} \qquad \text{for} \qquad x > c_{13}(E). \tag{3.6}$$

Taking logarithms in (3.5), then using (3.6), we get

$$\begin{aligned} L_2 &= l_2(1 - 1/l_1 + O_E(1/l_1 l_2)) \\ &= l_2(1 - 1/L_1 + O_E(1/L_1 L_2)) \qquad \text{for} \qquad x > c_{14}(E). \end{aligned}$$

It follows that

$$l_2 = L_2(1 + 1/L_1 + O_E(1/L_1 L_2)) \qquad \text{for} \quad x > c_{15}(E). \tag{3.7}$$

Substituting (3.7) in (3.5), replacing $O_E(1/l_1)$ by $O_E(1/L_1)$, and solving for l_1, we get

$$l_1 = L_1 + L_2 - \log \gamma(E) + L_2/L_1 + O_E(1/L_1) \qquad \text{for} \qquad x > c_{16}(E). \tag{3.8}$$

We now need to estimate $\theta(x;E)$ in terms of $\pi(x;E)$. We use the Stieltjes integral, then integrate by parts and combine with (1.7):

$$\begin{aligned} \theta(x;E) &= \int_1^x (\log t)\, d\pi(t;E) = \pi(x;E)\, l_1 - \int_2^x \frac{\gamma(E)}{\log t}\, dt \\ &\quad + O_E\left(\int_2^x \frac{dt}{(\log t)^2}\right) = \pi(x;E)\, l_1 - \gamma(E)(x/l_1) + O_E(x/l_1^2) \\ &= \pi(x;E)\, l_1 - \pi(x;E) + O_E(x/l_1^2) \end{aligned} \tag{3.9}$$

for $x \geqslant 2$. Now, (1.7) shows that

$$x/l_1^2 = O_E\left(\pi(x; E)/l_1\right) = O_E\left(\pi(x; E)/L_1\right) \qquad \text{for} \quad x > c_{17}(E).$$

Using this fact as well as (3.9) and (3.8), we get (3.4). Q.E.D.

Proof of Theorem 1.6: Write $E = \{p_1, p_2, p_3, \ldots\}$, where $p_1 < p_2 < p_3 < \ldots$. Define $n_r = p_1 p_2 \ldots p_r$ for $r \geqslant 1$. By (3.4),

$$\log n_r = \theta(p_r; E) = (r \log r)\left\{1 + \frac{\log_2 r}{\log r} - \frac{1 + \log \gamma(E)}{\log r} + \frac{\log_2 r}{(\log r)^2} + O_E\left(\frac{1}{(\log r)^2}\right)\right\} \tag{3.10}$$

for $r > c_{11}(E)$. Hence for $r > c_{18}(E)$,

$$\log_2 n_r = \log r + \log_2 r + \frac{\log_2 r}{\log r} + O_E\left(\frac{1}{\log r}\right). \tag{3.11}$$

If $r > c_{19}(E)$, then (3.10) and (3.11) yield

$$\log n_r = r\{\log_2 n_r - 1 - \log \gamma(E) + O_E(1/\log_2 n_r)\}. \tag{3.12}$$

If $r > c_{20}(E)$, we can solve (3.12) for r to get

$$\omega(n_r; E) = r = \frac{\log n_r}{\log_2 n_r}\left\{1 + \frac{1 + \log \gamma(E)}{\log_2 n_r} + O_E\left(\frac{1}{(\log_2 n_r)^2}\right)\right\}. \tag{3.13}$$

Now let n be any integer $\geqslant 3$, and write $\omega(n; E) = r$. Define

$$f(n, \alpha) = \frac{\log n}{\log_2 n} + \frac{\{1 + \log \gamma(E)\} \log n}{(\log_2 n)^2} + \alpha \frac{\log n}{(\log_2 n)^3}$$

for real α. For fixed positive α, $f(n, \alpha)$ increases with n for $n > c_{21}(\alpha, E)$. Thus if $r > c_{22}(E)$, it follows from (3.13) (since $n \geqslant n_r$) that

$$\omega(n; E) = \omega(n_r; E) \leqslant f\left(n_r, c_{23}(E)\right) \leqslant f\left(n, c_{23}(E)\right).$$

Now suppose that $0 \leqslant r = \omega(n; E) \leqslant c_{22}(E)$. If $n \geqslant c_{24}(E)$, then clearly

$$f\left(n, c_{23}(E)\right) \geqslant c_{22}(E) \geqslant \omega(n; E).$$

If $3 \leqslant n < c_{24}(E)$ and $c_{25}(E)$ is sufficiently large, then (since $\gamma(E) \leqslant 1$)

$$f\left(n, c_{25}(E)\right) \geqslant \frac{\log n}{(\log_2 n)^3}\left\{c_{25}(E) + \left(\log \gamma(E)\right) \log_2 c_{24}(E)\right\}$$
$$\geqslant c_{22}(E) \geqslant \omega(n; E).$$

It follows that if

$$c_{26}(E) = \max\{c_{23}(E), c_{25}(E)\},$$

then

$$\omega(n; E) \leqslant f(n, c_{26}(E))$$

for all $n \geqslant 3$. This proves (1.8), and (3.13) shows that equality holds in (1.8) for infinitely many n. Q.E.D.

For a more precise version of (1.8) when E is the set of all primes, see [12, p. 99].

Even a much weaker hypothesis than (1.7) implies that the maximum order of $\omega(n; E)$ is nearly $(\log n)(\log_2 n)^{-1}$. Specifically, suppose that there exist positive real numbers δ, x_0 such that

$$\pi(x; E) \geqslant x^{\delta} \qquad \text{for all} \quad x \geqslant x_0. \tag{3.14}$$

In the notation of the preceding proof, it is then clear that for $r \geqslant x_0$,

$$\log n_r = \theta(p_r; E) \geqslant \pi(p_r; E) - 1 \geqslant p_r^{\delta} - 1. \tag{3.15}$$

But trivially $\theta(p_r; E) \leqslant r \log p_r$, so

$$\omega(n_r; E) = r \geqslant (\log n_r)(\log p_r)^{-1},$$

and hence by (3.15),

$$\limsup_{n\to+\infty} \frac{\omega(n; E)\log_2 n}{\log n} \geqslant \delta. \tag{3.16}$$

Proof of Theorem 1.11: We use the method of Erdös and Nicolas [2], which we can refine and generalize by appealing to Lemma 3.3. As before, write

$$E = \{p_1, p_2, p_3, \ldots\},$$

where

$$p_1 < p_2 < p_3 < \ldots.$$

Assume that y satisfies (1.12) (where $c_2(E)$ is sufficiently large), take $r = [y] + 1$, and let $n_r = p_1 p_2 \ldots p_r$. There are exactly $[x/n_r]$ multiples n of n_r such that $n \leqslant x$, and for each such n we have $\omega(n; E) \geqslant r > y$. Hence

$$S(x, y; E, \omega) \geqslant [x/n_r]. \tag{3.17}$$

By (3.4),

$$\log n_r = r\{\log r + \log_2 r - 1 - \log\gamma(E) + O_E((\log_2 r)/\log r)\}$$

for $r > c_{11}(E)$. Define

$$g(t) = t\{\log t + \log_2 t - 1 - \log \gamma(E)\}$$

for real $t \geqslant 3$ and note that

$$0 < g'(t) = O_E(\log t) \qquad \text{for} \quad t \geqslant 3 .$$

By the mean-value theorem for derivatives,

$$g(r) = g(y) + O_E(\log y) ,$$

and hence

$$\log n_r = g(y) + O_E\bigl(y(\log_2 y)/\log y\bigr) \qquad \text{if} \quad y > c_{27}(E) . \tag{3.18}$$

In order to derive (1.13) from (3.17) and (3.18), we need to show that

$$[x/n_r] \gg x/n_r , \tag{3.19}$$

i.e., that $n_r \leqslant x$. For the remainder of this proof, write

$$l_k = \log_k x, \beta = 1 + \log \gamma(E) - \varepsilon ,$$

and

$$z = (l_1/l_2) + \beta\,(l_1/l_2^2) .$$

It follows from (1.12) that

$$y(\log_2 y)/\log y = O_E\bigl(l_1 l_3/l_2^2\bigr) .$$

Also, if $x > c_{28}(E, \varepsilon)$, then

$$\log z \leqslant l_2 - l_3 + (\beta/l_2) \leqslant l_2 - l_3 + (\varepsilon/2) ,$$
$$\log_2 z \leqslant l_3 .$$

It follows from these inequalities and (3.18) that if $x > c_1(E, \varepsilon)$ (sufficiently large) and (1.12) holds, then

$$\begin{aligned}
\log n_r &\leqslant g(z) + O_E\bigl(l_1 l_3/l_2^2\bigr) \\
&\leqslant (l_1/l_2)(1+\beta/l_2)\{l_2 - l_3 + (\varepsilon/2) + l_3 - (\beta+\varepsilon)\} \\
&\quad + O_E\bigl(l_1 l_3/l_2^2\bigr) \\
&= l_1(1-\varepsilon/2l_2) + O_{E,\varepsilon}\bigl(l_1 l_3/l_2^2\bigr) < l_1 ,
\end{aligned}$$

so $n_r < x$. Thus (3.19) holds, and (1.13) follows from (3.17) and (3.18). Q.E.D.

It is interesting to observe that a result somewhat like (1.13) can be deduced from a much weaker assumption than (1.7):

THEOREM 3.20. *Suppose there exist real numbers* $\delta > 0$, $x_0 \geqslant 2$ *such that* (3.14) *holds. If* $x \geqslant c_{29}(\delta)$ *and* $x_0 \leqslant y \leqslant \delta (\log x)(\log_2 x)^{-1}$, *then*

$$S(x, y; E, \omega) \gg x \exp\{-\delta^{-1}(y \log y + \log y + 2)\}.$$

Proof: In the notation of the preceding proof, (3.17) holds, and trivially $\log n_r \leqslant r \log p_r$. If $y \geqslant x_0$, then $p_r > r \geqslant x_0$ and $r = \pi(p_r; E) \geqslant p_r^\delta$, so

$$\begin{aligned} \log n_r \leqslant \delta^{-1} r \log r &\leqslant \delta^{-1}(y+1)(\log y + y^{-1}) \\ &\leqslant \delta^{-1}(y \log y + \log y + 2). \end{aligned} \tag{3.21}$$

But $\log y \leqslant \log_2 x - \log_3 x$, so $\log n_r < \log x$ if $x \geqslant c_{29}(\delta)$. Hence (3.19) holds, and the result follows from (3.17) and (3.21). Q.E.D.

§4. PROOFS OF THEOREM 1.14 AND RELATED RESULTS

We begin by quoting the following easy result from [13, pp. 689-690]:

LEMMA 4.1. *For* $x \geqslant 1$ *and* $z \geqslant 1$,

$$\sum_{n \leqslant x} z^{\omega(n; E)} \leqslant x \prod_{p \leqslant x,\, p \in E} \{1 + (z-1)p^{-1}\}.$$

To put this in a more convenient form, we prove

LEMMA 4.2. *If* $x \geqslant 1$ *and* $w \geqslant -2$, *then (cf.* (1.2)*)*

$$\prod_{p \leqslant x,\, p \in E} (1 + wp^{-1}) \leqslant e^{wE(x)}. \tag{4.3}$$

If $1 \leqslant w \leqslant x$, *then*

$$\prod_{p \leqslant x,\, p \in E} (1 + wp^{-1}) = \exp\{w(E(x) - E(w)) + O(w/\log(2w))\}. \tag{4.4}$$

Proof: (4.3) follows immediately from the inequalities

$$0 \leqslant 1 + wp^{-1} \leqslant \exp(wp^{-1}).$$

To get (4.4), we first write

$$\begin{aligned} \prod_{p \leqslant x,\, p \in E} (1 + wp^{-1}) &\leqslant \prod_{p \leqslant w} (2wp^{-1}) \cdot \prod_{w < p \leqslant x,\, p \in E} \exp(wp^{-1}) \\ &= \exp\{w(E(x) - E(w)) + \pi(w)\log(2w) - \theta(w)\}, \end{aligned}$$

where $\pi(w) = \sum_{p \leqslant w} 1$ and $\theta(w) = \sum_{p \leqslant w} \log p$. Since $\pi(t) \ll t/\log(2t)$ for $t \geqslant 1$, we have

$$\theta(w) = \int_1^w (\log t)\, d\pi(t) = \pi(w) \log w - \int_1^w \pi(t)\, t^{-1}\, dt$$
$$= \pi(w) \log w + O\big(w/\log(2w)\big),$$

and it follows that the right-hand side of (4.4) is an upper bound for the left-hand side. On the other hand, since $\log(1+y) = y + O(y^2)$ for $y > 0$, we have

$$\prod_{p \leqslant x,\, p \in E} (1+wp^{-1}) \geqslant \prod_{w < p \leqslant x,\, p \in E} \exp\{wp^{-1} + O(w^2 p^{-2})\}$$
$$= \exp\Big\{w\big(E(x) - E(w)\big) + O\Big(w^2 \sum_{p > w} p^{-2}\Big)\Big\}.$$

But

$$\sum_{p > w} p^{-2} = \int_w^{+\infty} t^{-2}\, d\pi(t) < 2 \int_w^{+\infty} t^{-3}\, \pi(t)\, dt \ll \big(w \log(2w)\big)^{-1},$$

and (4.4) follows. Q.E.D.

COROLLARY 4.5. *If* $x \geqslant 1$ *and* $z \geqslant 1$, *then*

$$\sum_{n \leqslant x} z^{\omega(n;E)} \leqslant x\, e^{(z-1)E(x)}. \tag{4.6}$$

If $1 \leqslant z \leqslant x$, *then*

$$\sum_{n \leqslant x} z^{\omega(n;E)}$$
$$\leqslant x \exp\{(z-1)\big(E(x) - E(z)\big) + c_{30} z/\log(2z)\}. \tag{4.7}$$

Note that if $1 \leqslant z < 2$, then (4.7) follows from (4.6).

THEOREM 4.8. *Let* $x \geqslant 1, v > 0, 1 \leqslant \alpha \leqslant x$. *Define* $\Lambda = \Lambda(x, v; E)$ *by* (1.22). *Then*

$$S(x, \alpha v; E, \omega) \leqslant x \exp\{(\alpha - 1 - \alpha \log \alpha) v - \alpha E(\alpha) + c_{31} \Lambda \alpha\}.$$

Proof: Suppose $1 \leqslant z \leqslant x$. Then

$$\sum_{n \leqslant x} z^{\omega(n;E)} \geqslant \sum_{n \leqslant x,\, \omega(n;E) > \alpha v} z^{\omega(n;E)} \geqslant z^{\alpha v} S(x, \alpha v; E, \omega).$$

Combining this result with (4.7), we get

$$S(x, \alpha v; E, \omega) \leqslant x \exp\{(z-1)(v + \Lambda) - zE(z) - \alpha v \log z$$
$$+ c_{32}\, z/\log(2z)\}. \tag{4.9}$$

In practice, we think of v as being a good approximation to $E(x)$, so that Λ is small compared to v. We want to minimize the right-hand side of (4.9) approximately, and for simplicity, we choose z so as to minimize the expression $(z-1)v - \alpha v \log z$, i.e., we take $z = \alpha$. With this value of z, we get the result from (4.9). Q.E.D.

LEMMA 4.10. *Suppose that there exists a real number* $\gamma(E) > 0$ *such that* (1.7) *holds. Then there is a real number* $\delta(E)$ *such that*

$$E(x) = \gamma(E) \log_2 x + \delta(E) + O_E(1/\log x) \quad \text{for} \quad x \geqslant 2. \tag{4.11}$$

Proof: Write

$$E(x) = \int_1^x t^{-1}\, d\pi(t; E),$$

integrate by parts, and use (1.7). Q.E.D.

From Theorem 4.8 and Lemma 4.10, we get

COROLLARY 4.12. *Suppose that there exists a real number* $\gamma(E) > 0$ *such that* (1.7) *holds. Let* $x \geqslant 3$, $2 \leqslant \alpha \leqslant x$. *Then*

$$S\left(x, \alpha\gamma(E) \log_2 x; E, \omega\right)$$
$$\leqslant x \exp\{(\alpha - 1 - \alpha \log \alpha)\gamma(E) \log_2 x - \alpha\gamma(E) \log_2 \alpha + c_{33}(E)\alpha\}.$$

Using (1.8), it is easy to show that Corollary 4.12 actually holds for all $\alpha \geqslant 2$, but it is also clear from (1.8) that

$$S\left(x, \alpha\gamma(E) \log_2 x; E, \omega\right) = 0$$

whenever α is somewhat greater than $(\log x)(\log_2 x)^{-2}$.

The upper bound given in Corollary 4.12 compares favorably with the theorem of Delange (Theorem 1.17 above), and our result is more general and holds for a much wider range of α. Our proof is also much simpler than Delange's. Unfortunately, our lower bound (1.13) is much smaller than the upper bound in Corollary 4.12.

Theorem 1.14 is proved in the same way as Theorem 4.8, but we use (4.6) instead of (4.7), apply Lemma 4.10, and take $z = y\left(\gamma(E) \log_2 x\right)^{-1}$.

We conclude this section by generalizing the Erdös-Nicolas result (1.10).

THEOREM 4.13. *Suppose that there exists a real number* $\gamma(E) > 0$ *such that* (1.7) *holds. Let* $\varepsilon > 0$, *and suppose that* $x \geqslant c_{34}(E, \varepsilon)$ *and*

$$(\log_2 x)^2 (\log x)^{-1} \leqslant \alpha \leqslant 1 + \{1 + \log \gamma(E) - \varepsilon\} (\log_2 x)^{-1}.$$

Then

$$x^{1-\alpha} \exp\left\{-c_{35}(E) \frac{\log x}{\log_2 x}\right\} \leqslant S\left(x, \alpha (\log x) (\log_2 x)^{-1}; E, \omega\right)$$

$$\leqslant x^{1-\alpha} \exp\left\{\frac{2\alpha (\log x) \log_3 x}{\log_2 x} + c_{36}(E) \frac{\log x}{\log_2 x}\right\}.$$

This can be obtained from Theorems 1.11 and 1.14 (take

$$y = \alpha (\log x) (\log_2 x)^{-1}$$

and use the inequalities

$$\log_2 y \leqslant \log_3 x, y \geqslant \log_2 x \geqslant \gamma(E) \log_2 x).$$

Theorem 4.13 should be compared with Theorem 1.6.

§5. PROOFS OF THEOREM 1.21 AND RELATED RESULTS

In estimating $S(x, y; E, \Omega)$ (defined by (1.1)), we do not need any assumption such as (1.7). Hence we emphasize that throughout the remainder of this paper, E is merely assumed to be any nonempty set of primes. (We shall sometimes assume explicitly that E has at least two members.) The smallest member of E will always be denoted by p_1 (and the smallest member of $E - \{p_1\}$, if it exists, by p_2). When x and v are positive real numbers, the function $\Lambda = \Lambda(x, v; E)$ is always defined by (1.22).

The subsequent work depends heavily on the following elementary lemma [13, p. 690]:

LEMMA 5.1. *If* $x > 0$ *and* $1 \leqslant z < p_1$, *then*

$$\sum_{n \leqslant x} z^{\Omega(n; E)} < p_1 (p_1 - z)^{-1} x\, e^{(z-1) E(x) + 4z}.$$

For the special case $E = P$, there is a recent paper of DeKoninck and Hensley [1] giving various estimates for $\sum\limits_{n \leqslant x}^{*} z^{\Omega(n)}$, where z is complex and $*$ indicates that the prime factors of n are restricted to lie in a certain range. DeKoninck and Hensley get sharp results, but their work is rather complicated and does not seem applicable to the problems discussed here.

If y is real and $z \geqslant 1$, then

$$\sum_{n \leqslant x} z^{\Omega(n; E)} \geqslant \sum_{n \leqslant x,\, \Omega(n; E) \geqslant y} z^{\Omega(n; E)}$$
$$\geqslant z^y \operatorname{card} \{n \leqslant x : \Omega(n; E) \geqslant y\}.$$

Hence Lemma 5.1 immediately yields

LEMMA 5.2. *If* $x > 0$, y *is real, and* $1 \leqslant z < p_1$, *then*

$$\operatorname{card} \{n \leqslant x : \Omega(n; E) \geqslant y\}$$
$$< p_1 (p_1 - z)^{-1} x \exp \{(z-1) E(x) - y \log z + 4z\}.$$

LEMMA 5.3. *Let* $x > 0, 0 < v \leqslant y < p_1 v$. *Then*

$$\operatorname{card} \{n \leqslant x : \Omega(n; E) \geqslant y\}$$
$$< c_{37}(p_1)(p_1 - y/v)^{-1} x \exp \{y - v - y \log (y/v) + p_1 \Lambda\}.$$

Proof: In Lemma 5.2, use the inequality $E(x) \leqslant v + \Lambda$ and take $z = y/v$ to get an approximate minimum. Q.E.D.

We observe in passing that Lemma 5.2 can also be used when $y \geqslant p_1 v$. In order to get a reasonably good result in this case by the same method, one needs to minimize the function

$$g(z) = (z-1) v - y \log z - \log (p_1 - z)$$

on the interval $1 \leqslant z < p_1$. Assuming that y is rather large, one can see with some computation that $g(z)$ is approximately minimized when

$$z = p_1 (1 - (2y)^{-1}),$$

and this z satisfies $1 \leqslant z < p_1$ whenever $y \geqslant 1$. With this value of z, Lemma 5.2 yields

$$\operatorname{card} \{n \leqslant x : \Omega(n; E) \geqslant y\} \leqslant c_{38}(p_1)\, y p_1^{-y} x\, e^{(p_1 - 1)v + p_1 \Lambda} \tag{5.4}$$

for $x > 0$, $y \geqslant 1$. When E is the set of all primes and $x \geqslant 3$, we can take $v = \log_2 x$, $\Lambda = O(1)$. Thus (5.4) is already sharper and more general than (1.20) (which is due to Erdös and Sárközy [3]). However, Theorem 1.18 shows that it may be of interest to take y as large as $(\log x)(\log p_1)^{-1}$, and we shall now prove that when y is relatively large, the factor y on the right-hand side of (5.4) can be replaced by a much smaller quantity.

LEMMA 5.5. *Write* $F = E - \{p_1\}$ *(if* F *is empty, we define* $\Omega(n; F) = 0$ *for all* n*). Let* $x > 0, y \geqslant 0$, *and let* $k = [y] + 1$. *For integers* a *with* $0 \leqslant a \leqslant k$, *define*

$$C_a = \{m \leqslant xp_1^{-a} : p_1 \nmid m \quad \text{and} \quad \Omega(m; F) \geqslant k - a\}.$$

Then

$$S(x, y; E, \Omega) = [xp_1^{-k}] + \sum_{a=0}^{k-1} \text{card } C_a .$$

Proof: For $0 \leqslant a \leqslant k$, define

$$B_a = \{n \leqslant x : p_1^a \parallel n \qquad \text{and} \qquad \Omega(np_1^{-a}; F) \geqslant k - a\}$$

(recall that $p_1^a \parallel n$ means $p_1^a \mid n$ and $p_1^{a+1} \nmid n$). It is easy to see that

$$\{n \leqslant x : \Omega(n; E) > y\} = \{n \leqslant x : p_1^k \mid n\} \cup \bigcup_{a=0}^{k-1} B_a .$$

Since the sets $\{n \leqslant x : p_1^k \mid n\}, B_0, B_1, ..., B_{k-1}$ are disjoint, we have

$$S(x, y; E, \Omega) = \text{card } \{n \leqslant x : p_1^k \mid n\} + \sum_{a=0}^{k-1} \text{card } B_a .$$

But the mapping $n \mapsto np_1^{-a}$ establishes a one-to-one correspondence between B_a and C_a, so the result follows. Q.E.D.

Proof of Theorem 1.21: If $E = \{p_1\}$, then by Lemma 5.5,

$$S(x, y; E, \Omega) \leqslant xp_1^{-y},$$

and (1.23) follows. Thus we may assume that $F = E - \{p_1\}$ is not empty. Let p_2 be the smallest member of F, and let $k = [y] + 1$. By Lemma 5.5,

$$S(x, y; E, \Omega) = [xp_1^{-k}] + \sum_{a=1}^{k} \text{card } C_{k-a} . \tag{5.6}$$

To estimate

$$\text{card } C_{k-a} = \text{card } \{m \leqslant xp_1^{a-k} : p_1 \nmid m \quad \text{and} \quad \Omega(m; F) \geqslant a\}$$

from above, we apply Lemma 5.2 (with E replaced by F and p_1 by p_2). Since

$$F(xp_1^{a-k}) \leqslant F(x) \leqslant E(x) \leqslant v + \Lambda ,$$

we obtain

$$\begin{aligned} &\text{card } C_{k-a} \\ &< p_2 (p_2 - z)^{-1} \, xp_1^{a-k} \exp \{(z-1)(v+\Lambda) - a \log z + 4z\} \\ &= H(a, z) , \end{aligned} \tag{5.7}$$

say, and this holds for each integer a $(1 \leqslant a \leqslant k)$ and each real z with $1 \leqslant z < p_2$. In applying (5.7), we are free to choose z to depend on a. Write $Q = \max \{k, p_1 v\}$, and for each a $(1 \leqslant a \leqslant Q)$, let z_a be any real number satisfying $1 \leqslant z_a < p_2$. Then by (5.6) and (5.7),

$$\begin{aligned} S(x, y; E, \Omega) &\leqslant xp_1^{-k} + \sum_{a=1}^{k} H(a, z_a) \\ &\leqslant xp_1^{-k} + \sum_{1 \leqslant a \leqslant v} H(a, z_a) + \sum_{v < a \leqslant p_1 v} H(a, z_a) \\ &\quad + \sum_{p_1 v < a \leqslant Q} H(a, z_a) . \end{aligned} \tag{5.8}$$

For $1 \leqslant a \leqslant v$, take $z_a = 1$. With this choice, we have

$$\begin{aligned} \sum_{1 \leqslant a \leqslant v} H(a, z_a) &\ll xp_1^{-k} \sum_{1 \leqslant a \leqslant v} p_1^a \ll xp_1^{-y+v} \\ &\ll xp_1^{-y} e^{(p_1 - 1) v} . \end{aligned} \tag{5.9}$$

For $v < a \leqslant p_1 v$, the quantity $(z-1) v - a \log z$ in (5.7) is minimized by taking $z = a/v = z_a$. With this choice of z_a, we have $1 < z_a \leqslant p_1$ and

$$p_2 (p_2 - z_a)^{-1} \leqslant p_2 (p_2 - p_1)^{-1} \leqslant 1 + p_1 ,$$

so

$$H(a, z_a) \leqslant c_{39}(p_1) \, xp_1^{a-k} e^{(p_1 - 1) \Lambda} (v^a e^{-v} / a^a e^{-a}) .$$

By Stirling's formula, $a^a e^{-a} \gg a! \, a^{-1/2}$, so we get

$$\begin{aligned} \sum_{v < a \leqslant p_1 v} H(a, z_a) &\leqslant c_{40}(p_1) \, xp_1^{-y} v^{1/2} e^{-v + p_1 \Lambda} \sum_{v < a \leqslant p_1 v} \frac{(p_1 v)^a}{a!} \\ &\leqslant c_{40}(p_1) \, xp_1^{-y} v^{1/2} e^{(p_1 - 1) v + p_1 \Lambda} . \end{aligned} \tag{5.10}$$

For $p_1 v < a \leqslant Q$, we let all the numbers z_a have the same value $p_1(1+\theta)$, where θ is a real number about which we assume only that $0 < \theta < p_2 p_1^{-1} - 1$ (the last inequality being needed in order to have $z_a < p_2$). With this choice of z_a, (5.7) yields

$$\sum_{p_1 v < a \leqslant Q} H(a, z_a)$$

$$\leqslant p_2 \{p_2 - p_1(1+\theta)\}^{-1} x p_1^{-k} \exp\{(p_1 - 1 + p_1\theta)(v+\Lambda) + 4p_1(1+\theta)\}$$

$$\times \sum_{p_1 v < a \leqslant Q} (1+\theta)^{-a}. \tag{5.11}$$

The last sum on the right does not exceed

$$\sum_{a > p_1 v} (1+\theta)^{-a} < (1+\theta)\,\theta^{-1}(1+\theta)^{-p_1 v}. \tag{5.12}$$

After combining this estimate with (5.11), we would like to minimize the contribution of the essential terms $e^{p_1\theta v}\,\theta^{-1}(1+\theta)^{-p_1 v}$. Since

$$\log(1+\theta) \geqslant \theta - \theta^2/2 \qquad \text{for} \quad \theta \geqslant 0, \tag{5.13}$$

we have

$$p_1\theta v - \log\theta - p_1 v \log(1+\theta) \leqslant -\log\theta + p_1 v\theta^2/2,$$

and here the right-hand side would be minimized by taking θ to be $(p_1 v)^{-1/2}$. However, we must also choose $\theta < p_2 p_1^{-1} - 1$ (so that $z_a < p_2$). If we take

$$\theta = (2p_1 v^{1/2})^{-1}, \tag{5.14}$$

then because of our assumption that $v \geqslant 1$, we have

$$\theta \leqslant (2p_1)^{-1} < p_2 p_1^{-1} - 1.$$

Combining (5.11), (5.12), (5.13), and (5.14), and observing that

$$\begin{aligned} p_2\{p_2 - p_1(1+\theta)\}^{-1} &\leqslant p_2(p_2 - p_1 - 1/2)^{-1} \\ &= 1 + (p_1 + 1/2)(p_2 - p_1 - 1/2)^{-1} \leqslant c_{41}(p_1), \end{aligned}$$

we obtain finally

$$\sum_{p_1 v < a \leqslant Q} H(a, z_a) \leqslant c_{42}(p_1)\, x p_1^{-v} v^{1/2} e^{(p_1 - 1)v + p_1\Lambda}. \tag{5.15}$$

The theorem now follows from (5.8), (5.9), (5.10), and (5.15). Q.E.D.

Since

$$E(x) \leqslant \sum_{p \leqslant x} p^{-1} = \log_2 x + O(1) \qquad \text{for} \quad x \geqslant 2,$$

one would always want to choose $v \leqslant \log_2 x$. Thus (1.23) is superior to (5.4) whenever $y \geqslant (\log_2 x)^{1/2}$. Furthermore, consideration of derivatives shows that

$$y - v - y \log(y/v) \leqslant (p_1 - 1)v - y \log p_1 \qquad \text{for} \qquad 0 < v \leqslant y \leqslant p_1 v,$$

and hence Lemma 5.3 is superior to Theorem 1.21 whenever

$$1 \leqslant v \leqslant y \leqslant p_1 v - v^{1/2}.$$

REFERENCES

[1] DEKONINCK, J.-M. and D. HENSLEY. Sums taken over $n \leqslant x$ with prime factors $\leqslant y$ of $z^{\Omega(n)}$, and their derivatives with respect to z. *J. Indian Math. Soc. (N.S.) 42* (1978), 353-365.

[2] ERDÖS, P. et J.-L. NICOLAS. Sur la fonction: nombre de facteurs premiers de N. *Ens. Math. 27* (1981), 3-30.

[3] ERDÖS, P. and A. SÁRKÖZY. On the number of prime factors of integers. *Acta Sci. Math. (Szeged) 42* (1980), 237-246.

[4] HALÁSZ, G. Remarks to my paper "On the distribution of additive and the mean values of multiplicative arithmetic functions". *Acta Math. Acad. Sci. Hungar. 23* (1972), 425-432.

[5] HARDY, G. H. and S. RAMANUJAN. The normal number of prime factors of a number n. *Quart. J. Math. 48* (1917), 76-92.

[6] HARDY, G. H. and E. M. WRIGHT. *An introduction to the theory of numbers.* 3rd ed., Oxford Univ. Press, Oxford, 1954.

[7] KOLESNIK, G. and E. G. STRAUS. On the distribution of integers with a given number of prime factors. *Acta Arith. 37* (1980), 181-199.

[8] KUBILIUS, J. P. *Probabilistic methods in the theory of numbers.* Amer. Math. Soc. Translations of Mathematical Monographs, vol. 11, Providence, R.I., 1964.

[9] —— On large deviations of additive arithmetic functions. *Trudy Mat. Inst. Steklov. 128* (1972), 163-171 (= *Proc. Steklov Inst. Math. 128* (1972), 191-201).

[10] LAURINČIKAS, A. On large deviations of arithmetic functions. *Litovsk. Mat. Sb. 16* (1976), No. 1, 159-171 (= *Lithuanian Math. Trans. 16* (1976), 97-104).

[11] NEWMAN, M. Isometric circles of congruence groups. *Amer. J. Math. 91* (1969), 648-656.

[12] NORTON, K. K. Numbers with small prime factors, and the least k-th power non-residue. *Mem. Amer. Math. Soc. 106* (1971).

[13] —— On the number of restricted prime factors of an integer. I. *Illinois J. Math. 20* (1976), 681-705.

[14] —— On the number of restricted prime factors of an integer. II. *Acta Math. 143* (1979), 9-38.

[15] RAMANUJAN, S. *Collected papers.* Cambridge Univ. Press, Cambridge, 1927. (Reprinted by Chelsea, New York, 1962.)

[16] SÁRKÖZY, A. Remarks on a paper of G. Halász. *Period. Math. Hungar. 8* (1977), 135-150.

[17] SELBERG, A. Note on a paper by L. G. Sathe. *J. Indian Math. Soc. (N.S.) 18* (1954), 83-87.

[18] TURÁN, P. On a theorem of Hardy and Ramanujan. *J. London Math. Soc. 9* (1934), 274-276.

[19] —— Über einige Verallgemeinerungen eines Satzes von Hardy und Ramanujan. *J. London Math. Soc. 11* (1936), 125-133.

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