§3. Proofs of Theorems 1.6 and 1.11, and related results

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **28 (1982)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **21.07.2024**

Empty sums mean 0, empty products 1, and we define $0^0 = 1$. The notation

$$x_1 \cdots x_m / y_1 \cdots y_n$$

is sometimes used instead of

$$(x_1 \cdots x_m)(y_1 \cdots y_n)^{-1}.$$

Throughout this paper, E denotes a nonempty set of primes, to be regarded as quite arbitrary unless further assumptions are stated. $E(x)$ is always defined by (1.2). p_1 always means the smallest member of E, and if

$$E - \{p_1\} = \{p : p \in E \quad \text{and} \quad p \neq p_1\}$$

is not empty, then p_2 denotes the smallest member of $E - \{p_1\}$. When x and v are positive, the function $\Lambda = \Lambda(x, v; E)$ is always defined by (1.22).

§3. Proofs of Theorems 1.6 and 1.11, and related results

Before proving (1.8), we observe that a similar but weaker inequality has a very simple proof. For if $y > 1$, then

$$\log n \geqslant \sum_{p \mid n} \log p \geqslant \sum_{p \mid n,\, p \geqslant y} \log p \geqslant (\log y) \sum_{p \mid n,\, p \geqslant y} 1,$$

and hence

$$\omega(n) = \sum_{p \mid n,\, p < y} 1 + \sum_{p \mid n,\, p \geqslant y} 1 \leqslant y + (\log n)(\log y)^{-1}.$$

The right-hand side is approximately minimized by taking

$$y = (\log n)(\log_2 n)^{-2},$$

and we obtain

$$\omega(n) \leqslant \frac{\log n}{\log_2 n} \left\{ 1 + O\left(\frac{\log_3 n}{\log_2 n} \right) \right\} \quad \text{for} \quad n \geqslant 16\,(> e^e). \tag{3.1}$$

Another simple proof of (3.1) can be based on Newman's observation [11, p. 652] that if $\omega(n) = r$, then $n \geqslant r!$.

To get the sharper inequality (1.8), it seems to be necessary to use an assumption such as (1.7) about the distribution of E. First we need a lemma relating $\pi(x; E)$ (defined by (1.4)) and

$$\theta(x; E) = \sum_{p \leqslant x,\, p \in E} \log p. \tag{3.2}$$

LEMMA 3.3. *Suppose that there exists a real number* $\gamma(E) > 0$ *such that* (1.7) *holds. Then for* $x > c_{11}(E)$,

$$\theta(x;E) = \pi(x;E)\left\{\log \pi(x;E) + \log_2 \pi(x;E) - 1 - \log \gamma(E) + \frac{\log_2 \pi(x;E)}{\log \pi(x;E)} + O_E\left(\frac{1}{\log \pi(x;E)}\right)\right\}. \tag{3.4}$$

Proof: For notational simplicity, we write $l_r = \log_r x$, $L_r = \log_r \pi(x;E)$ whenever these are defined. First note that for $x > c_{12}(E)$, (1.7) implies

$$L_1 = l_1 - l_2 + \log \gamma(E) + O_E(1/l_1). \tag{3.5}$$

In particular, $L_1 \sim l_1$ and $L_2 \sim l_2$ as $x \to +\infty$, so for $x > c_{13}(E)$, (3.5) implies

$$L_1 = l_1\{1 + O_E(L_2/L_1)\},$$

and multiplication by $(L_1 l_1)^{-1}$ yields

$$l_1^{-1} = L_1^{-1}\{1 + O_E(L_2/L_1)\} \qquad \text{for} \qquad x > c_{13}(E). \tag{3.6}$$

Taking logarithms in (3.5), then using (3.6), we get

$$\begin{aligned} L_2 &= l_2(1 - 1/l_1 + O_E(1/l_1 l_2)) \\ &= l_2(1 - 1/L_1 + O_E(1/L_1 L_2)) \qquad \text{for} \qquad x > c_{14}(E). \end{aligned}$$

It follows that

$$l_2 = L_2(1 + 1/L_1 + O_E(1/L_1 L_2)) \qquad \text{for} \quad x > c_{15}(E). \tag{3.7}$$

Substituting (3.7) in (3.5), replacing $O_E(1/l_1)$ by $O_E(1/L_1)$, and solving for l_1, we get

$$l_1 = L_1 + L_2 - \log \gamma(E) + L_2/L_1 + O_E(1/L_1) \qquad \text{for} \qquad x > c_{16}(E). \tag{3.8}$$

We now need to estimate $\theta(x;E)$ in terms of $\pi(x;E)$. We use the Stieltjes integral, then integrate by parts and combine with (1.7):

$$\begin{aligned} \theta(x;E) &= \int_1^x (\log t)\, d\pi(t;E) = \pi(x;E)\, l_1 - \int_2^x \frac{\gamma(E)}{\log t}\, dt \\ &\quad + O_E\left(\int_2^x \frac{dt}{(\log t)^2}\right) = \pi(x;E)\, l_1 - \gamma(E)(x/l_1) + O_E(x/l_1^2) \\ &= \pi(x;E)\, l_1 - \pi(x;E) + O_E(x/l_1^2) \end{aligned} \tag{3.9}$$

for $x \geqslant 2$. Now, (1.7) shows that

$$x/l_1^2 = O_E(\pi(x; E)/l_1) = O_E(\pi(x; E)/L_1) \quad \text{for} \quad x > c_{17}(E).$$

Using this fact as well as (3.9) and (3.8), we get (3.4). Q.E.D.

Proof of Theorem 1.6: Write $E = \{p_1, p_2, p_3, \ldots\}$, where $p_1 < p_2 < p_3 < \ldots$. Define $n_r = p_1 p_2 \ldots p_r$ for $r \geqslant 1$. By (3.4),

$$\log n_r = \theta(p_r; E) = (r \log r)\left\{1 + \frac{\log_2 r}{\log r} - \frac{1 + \log \gamma(E)}{\log r} + \frac{\log_2 r}{(\log r)^2} + O_E\left(\frac{1}{(\log r)^2}\right)\right\} \tag{3.10}$$

for $r > c_{11}(E)$. Hence for $r > c_{18}(E)$,

$$\log_2 n_r = \log r + \log_2 r + \frac{\log_2 r}{\log r} + O_E\left(\frac{1}{\log r}\right). \tag{3.11}$$

If $r > c_{19}(E)$, then (3.10) and (3.11) yield

$$\log n_r = r\{\log_2 n_r - 1 - \log \gamma(E) + O_E(1/\log_2 n_r)\}. \tag{3.12}$$

If $r > c_{20}(E)$, we can solve (3.12) for r to get

$$\omega(n_r; E) = r = \frac{\log n_r}{\log_2 n_r}\left\{1 + \frac{1 + \log \gamma(E)}{\log_2 n_r} + O_E\left(\frac{1}{(\log_2 n_r)^2}\right)\right\}. \tag{3.13}$$

Now let n be any integer $\geqslant 3$, and write $\omega(n; E) = r$. Define

$$f(n, \alpha) = \frac{\log n}{\log_2 n} + \frac{\{1 + \log \gamma(E)\} \log n}{(\log_2 n)^2} + \alpha \frac{\log n}{(\log_2 n)^3}$$

for real α. For fixed positive α, $f(n, \alpha)$ increases with n for $n > c_{21}(\alpha, E)$. Thus if $r > c_{22}(E)$, it follows from (3.13) (since $n \geqslant n_r$) that

$$\omega(n; E) = \omega(n_r; E) \leqslant f(n_r, c_{23}(E)) \leqslant f(n, c_{23}(E)).$$

Now suppose that $0 \leqslant r = \omega(n; E) \leqslant c_{22}(E)$. If $n \geqslant c_{24}(E)$, then clearly

$$f(n, c_{23}(E)) \geqslant c_{22}(E) \geqslant \omega(n; E).$$

If $3 \leqslant n < c_{24}(E)$ and $c_{25}(E)$ is sufficiently large, then (since $\gamma(E) \leqslant 1$)

$$f(n, c_{25}(E)) \geqslant \frac{\log n}{(\log_2 n)^3}\left\{c_{25}(E) + (\log \gamma(E)) \log_2 c_{24}(E)\right\}$$
$$\geqslant c_{22}(E) \geqslant \omega(n; E).$$

It follows that if

$$c_{26}(E) = \max\{c_{23}(E), c_{25}(E)\},$$

then

$$\omega(n; E) \leqslant f(n, c_{26}(E))$$

for all $n \geqslant 3$. This proves (1.8), and (3.13) shows that equality holds in (1.8) for infinitely many n. Q.E.D.

For a more precise version of (1.8) when E is the set of all primes, see [12, p. 99].

Even a much weaker hypothesis than (1.7) implies that the maximum order of $\omega(n; E)$ is nearly $(\log n)(\log_2 n)^{-1}$. Specifically, suppose that there exist positive real numbers δ, x_0 such that

$$\pi(x; E) \geqslant x^{\delta} \qquad \text{for all} \quad x \geqslant x_0. \tag{3.14}$$

In the notation of the preceding proof, it is then clear that for $r \geqslant x_0$,

$$\log n_r = \theta(p_r; E) \geqslant \pi(p_r; E) - 1 \geqslant p_r^{\delta} - 1. \tag{3.15}$$

But trivially $\theta(p_r; E) \leqslant r \log p_r$, so

$$\omega(n_r; E) = r \geqslant (\log n_r)(\log p_r)^{-1},$$

and hence by (3.15),

$$\limsup_{n \to +\infty} \frac{\omega(n; E)\log_2 n}{\log n} \geqslant \delta. \tag{3.16}$$

Proof of Theorem 1.11: We use the method of Erdös and Nicolas [2], which we can refine and generalize by appealing to Lemma 3.3. As before, write

$$E = \{p_1, p_2, p_3, \ldots\},$$

where

$$p_1 < p_2 < p_3 < \ldots.$$

Assume that y satisfies (1.12) (where $c_2(E)$ is sufficiently large), take $r = [y] + 1$, and let $n_r = p_1 p_2 \ldots p_r$. There are exactly $[x/n_r]$ multiples n of n_r such that $n \leqslant x$, and for each such n we have $\omega(n; E) \geqslant r > y$. Hence

$$S(x, y; E, \omega) \geqslant [x/n_r]. \tag{3.17}$$

By (3.4),

$$\log n_r = r\{\log r + \log_2 r - 1 - \log\gamma(E) + O_E((\log_2 r)/\log r)\}$$

for $r > c_{11}(E)$. Define

$$g(t) = t\{\log t + \log_2 t - 1 - \log \gamma(E)\}$$

for real $t \geqslant 3$ and note that

$$0 < g'(t) = O_E(\log t) \qquad \text{for} \quad t \geqslant 3 .$$

By the mean-value theorem for derivatives,

$$g(r) = g(y) + O_E(\log y),$$

and hence

$$\log n_r = g(y) + O_E\big(y(\log_2 y)/\log y\big) \qquad \text{if} \quad y > c_{27}(E) . \tag{3.18}$$

In order to derive (1.13) from (3.17) and (3.18), we need to show that

$$[x/n_r] \gg x/n_r , \tag{3.19}$$

i.e., that $n_r \leqslant x$. For the remainder of this proof, write

$$l_k = \log_k x, \beta = 1 + \log \gamma(E) - \varepsilon ,$$

and

$$z = (l_1/l_2) + \beta(l_1/l_2^2) .$$

It follows from (1.12) that

$$y(\log_2 y)/\log y = O_E\big(l_1 l_3/l_2^2\big) .$$

Also, if $x > c_{28}(E, \varepsilon)$, then

$$\log z \leqslant l_2 - l_3 + (\beta/l_2) \leqslant l_2 - l_3 + (\varepsilon/2) ,$$
$$\log_2 z \leqslant l_3 .$$

It follows from these inequalities and (3.18) that if $x > c_1(E, \varepsilon)$ (sufficiently large) and (1.12) holds, then

$$\begin{aligned} \log n_r &\leqslant g(z) + O_E\big(l_1 l_3/l_2^2\big) \\ &\leqslant (l_1/l_2)(1+\beta/l_2)\{l_2 - l_3 + (\varepsilon/2) + l_3 - (\beta+\varepsilon)\} \\ &\quad + O_E\big(l_1 l_3/l_2^2\big) \\ &= l_1(1-\varepsilon/2l_2) + O_{E,\varepsilon}\big(l_1 l_3/l_2^2\big) < l_1 , \end{aligned}$$

so $n_r < x$. Thus (3.19) holds, and (1.13) follows from (3.17) and (3.18). Q.E.D.

It is interesting to observe that a result somewhat like (1.13) can be deduced from a much weaker assumption than (1.7):

THEOREM 3.20. *Suppose there exist real numbers* $\delta > 0$, $x_0 \geqslant 2$ *such that* (3.14) *holds. If* $x \geqslant c_{29}(\delta)$ *and* $x_0 \leqslant y \leqslant \delta(\log x)(\log_2 x)^{-1}$, *then*

$$S(x, y; E, \omega) \gg x \exp\{-\delta^{-1}(y \log y + \log y + 2)\}.$$

Proof: In the notation of the preceding proof, (3.17) holds, and trivially $\log n_r \leqslant r \log p_r$. If $y \geqslant x_0$, then $p_r > r \geqslant x_0$ and $r = \pi(p_r; E) \geqslant p_r^\delta$, so

$$\begin{aligned}\log n_r \leqslant \delta^{-1} r \log r &\leqslant \delta^{-1}(y+1)(\log y + y^{-1}) \\ &\leqslant \delta^{-1}(y \log y + \log y + 2). \qquad (3.21)\end{aligned}$$

But $\log y \leqslant \log_2 x - \log_3 x$, so $\log n_r < \log x$ if $x \geqslant c_{29}(\delta)$. Hence (3.19) holds, and the result follows from (3.17) and (3.21). Q.E.D.

§4. PROOFS OF THEOREM 1.14 AND RELATED RESULTS

We begin by quoting the following easy result from [13, pp. 689-690]:

LEMMA 4.1. *For* $x \geqslant 1$ *and* $z \geqslant 1$,

$$\sum_{n \leqslant x} z^{\omega(n; E)} \leqslant x \prod_{p \leqslant x,\, p \in E} \{1 + (z-1)p^{-1}\}.$$

To put this in a more convenient form, we prove

LEMMA 4.2. *If* $x \geqslant 1$ *and* $w \geqslant -2$, *then (cf.* (1.2))

$$\prod_{p \leqslant x,\, p \in E} (1 + wp^{-1}) \leqslant e^{wE(x)}. \qquad (4.3)$$

If $1 \leqslant w \leqslant x$, *then*

$$\begin{aligned}&\prod_{p \leqslant x,\, p \in E} (1 + wp^{-1}) \\ &= \exp\{w(E(x) - E(w)) + O(w/\log(2w))\}. \qquad (4.4)\end{aligned}$$

Proof: (4.3) follows immediately from the inequalities

$$0 \leqslant 1 + wp^{-1} \leqslant \exp(wp^{-1}).$$

To get (4.4), we first write

$$\begin{aligned}\prod_{p \leqslant x,\, p \in E} (1 + wp^{-1}) &\leqslant \prod_{p \leqslant w} (2wp^{-1}) \cdot \prod_{w < p \leqslant x,\, p \in E} \exp(wp^{-1}) \\ &= \exp\{w(E(x) - E(w)) + \pi(w)\log(2w) - \theta(w)\},\end{aligned}$$