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Kapitel: §5. Proofs of Theorem 1.21 and related results

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THEOREM 4.13. *Suppose that there exists a real number* $\gamma(E) > 0$ *such that* (1.7) *holds. Let* $\varepsilon > 0$, *and suppose that* $x \geqslant c_{34}(E, \varepsilon)$ *and*

$$(\log_2 x)^2 (\log x)^{-1} \leqslant \alpha \leqslant 1 + \{1 + \log \gamma(E) - \varepsilon\} (\log_2 x)^{-1}.$$

Then

$$x^{1-\alpha} \exp\left\{-c_{35}(E) \frac{\log x}{\log_2 x}\right\} \leqslant S\left(x, \alpha (\log x)(\log_2 x)^{-1}; E, \omega\right)$$

$$\leqslant x^{1-\alpha} \exp\left\{\frac{2\alpha (\log x) \log_3 x}{\log_2 x} + c_{36}(E) \frac{\log x}{\log_2 x}\right\}.$$

This can be obtained from Theorems 1.11 and 1.14 (take

$$y = \alpha (\log x)(\log_2 x)^{-1}$$

and use the inequalities

$$\log_2 y \leqslant \log_3 x, y \geqslant \log_2 x \geqslant \gamma(E) \log_2 x).$$

Theorem 4.13 should be compared with Theorem 1.6.

§5. PROOFS OF THEOREM 1.21 AND RELATED RESULTS

In estimating $S(x, y; E, \Omega)$ (defined by (1.1)), we do not need any assumption such as (1.7). Hence we emphasize that throughout the remainder of this paper, E is merely assumed to be any nonempty set of primes. (We shall sometimes assume explicitly that E has at least two members.) The smallest member of E will always be denoted by p_1 (and the smallest member of $E - \{p_1\}$, if it exists, by p_2). When x and v are positive real numbers, the function $\Lambda = \Lambda(x, v; E)$ is always defined by (1.22).

The subsequent work depends heavily on the following elementary lemma [13, p. 690]:

LEMMA 5.1. *If* $x > 0$ *and* $1 \leqslant z < p_1$, *then*

$$\sum_{n \leqslant x} z^{\Omega(n; E)} < p_1 (p_1 - z)^{-1} x\, e^{(z-1) E(x) + 4z}.$$

For the special case $E = P$, there is a recent paper of DeKoninck and Hensley [1] giving various estimates for $\sum_{n \leqslant x}^{*} z^{\Omega(n)}$, where z is complex and $*$ indicates that the prime factors of n are restricted to lie in a certain range. DeKoninck and Hensley get sharp results, but their work is rather complicated and does not seem applicable to the problems discussed here.

If y is real and $z \geqslant 1$, then

$$\sum_{n \leqslant x} z^{\Omega(n; E)} \geqslant \sum_{n \leqslant x,\, \Omega(n;\, E) \geqslant y} z^{\Omega(n; E)}$$
$$\geqslant z^y \operatorname{card} \{n \leqslant x : \Omega(n; E) \geqslant y\}\,.$$

Hence Lemma 5.1 immediately yields

LEMMA 5.2. *If* $x > 0, y$ *is real, and* $1 \leqslant z < p_1$, *then*

$$\operatorname{card} \{n \leqslant x : \Omega(n; E) \geqslant y\}$$
$$< p_1 (p_1 - z)^{-1} x \exp \{(z-1) E(x) - y \log z + 4z\}\,.$$

LEMMA 5.3. *Let* $x > 0, 0 < v \leqslant y < p_1 v$. *Then*

$$\operatorname{card} \{n \leqslant x : \Omega(n; E) \geqslant y\}$$
$$< c_{37}(p_1)(p_1 - y/v)^{-1} x \exp \{y - v - y \log(y/v) + p_1 \Lambda\}\,.$$

Proof: In Lemma 5.2, use the inequality $E(x) \leqslant v + \Lambda$ and take $z = y/v$ to get an approximate minimum. Q.E.D.

We observe in passing that Lemma 5.2 can also be used when $y \geqslant p_1 v$. In order to get a reasonably good result in this case by the same method, one needs to minimize the function

$$g(z) = (z-1)v - y \log z - \log(p_1 - z)$$

on the interval $1 \leqslant z < p_1$. Assuming that y is rather large, one can see with some computation that $g(z)$ is approximately minimized when

$$z = p_1 \left(1 - (2y)^{-1}\right),$$

and this z satisfies $1 \leqslant z < p_1$ whenever $y \geqslant 1$. With this value of z, Lemma 5.2 yields

$$\operatorname{card} \{n \leqslant x : \Omega(n; E) \geqslant y\} \leqslant c_{38}(p_1)\, y p_1^{-y} x\, e^{(p_1 - 1)v + p_1 \Lambda} \tag{5.4}$$

for $x > 0$, $y \geqslant 1$. When E is the set of all primes and $x \geqslant 3$, we can take $v = \log_2 x$, $\Lambda = O(1)$. Thus (5.4) is already sharper and more general than (1.20) (which is due to Erdös and Sárközy [3]). However, Theorem 1.18 shows that it may be of interest to take y as large as $(\log x)(\log p_1)^{-1}$, and we shall now prove that when y is relatively large, the factor y on the right-hand side of (5.4) can be replaced by a much smaller quantity.

LEMMA 5.5. *Write* $F = E - \{p_1\}$ *(if* F *is empty, we define* $\Omega(n; F) = 0$ *for all* n*). Let* $x > 0, y \geqslant 0,$ *and let* $k = [y] + 1$. *For integers* a *with* $0 \leqslant a \leqslant k$, *define*

$$C_a = \{m \leqslant xp_1^{-a} : p_1 \nmid m \quad \text{and} \quad \Omega(m; F) \geqslant k - a\}.$$

Then

$$S(x, y; E, \Omega) = [xp_1^{-k}] + \sum_{a=0}^{k-1} \text{card } C_a .$$

Proof: For $0 \leqslant a \leqslant k$, define

$$B_a = \{n \leqslant x : p_1^a \parallel n \qquad \text{and} \qquad \Omega(np_1^{-a}; F) \geqslant k - a\}$$

(recall that $p_1^a \parallel n$ means $p_1^a \mid n$ and $p_1^{a+1} \nmid n$). It is easy to see that

$$\{n \leqslant x : \Omega(n; E) > y\} = \{n \leqslant x : p_1^k \mid n\} \cup \bigcup_{a=0}^{k-1} B_a .$$

Since the sets $\{n \leqslant x : p_1^k \mid n\}, B_0, B_1, \ldots, B_{k-1}$ are disjoint, we have

$$S(x, y; E, \Omega) = \text{card } \{n \leqslant x : p_1^k \mid n\} + \sum_{a=0}^{k-1} \text{card } B_a .$$

But the mapping $n \mapsto np_1^{-a}$ establishes a one-to-one correspondence between B_a and C_a, so the result follows. Q.E.D.

Proof of Theorem 1.21: If $E = \{p_1\}$, then by Lemma 5.5,

$$S(x, y; E, \Omega) \leqslant xp_1^{-y},$$

and (1.23) follows. Thus we may assume that $F = E - \{p_1\}$ is not empty. Let p_2 be the smallest member of F, and let $k = [y] + 1$. By Lemma 5.5,

$$S(x, y; E, \Omega) = [xp_1^{-k}] + \sum_{a=1}^{k} \text{card } C_{k-a} . \tag{5.6}$$

To estimate

$$\text{card } C_{k-a} = \text{card } \{m \leqslant xp_1^{a-k} : p_1 \nmid m \quad \text{and} \quad \Omega(m; F) \geqslant a\}$$

from above, we apply Lemma 5.2 (with E replaced by F and p_1 by p_2). Since

$$F(xp_1^{a-k}) \leqslant F(x) \leqslant E(x) \leqslant v + \Lambda,$$

we obtain

$$\begin{aligned} &\text{card } C_{k-a} \\ &< p_2 (p_2 - z)^{-1} xp_1^{a-k} \exp\{(z-1)(v+\Lambda) - a \log z + 4z\} \\ &= H(a, z), \end{aligned} \tag{5.7}$$

say, and this holds for each integer a $(1 \leqslant a \leqslant k)$ and each real z with $1 \leqslant z < p_2$. In applying (5.7), we are free to choose z to depend on a. Write $Q = \max\{k, p_1 v\}$, and for each a $(1 \leqslant a \leqslant Q)$, let z_a be any real number satisfying $1 \leqslant z_a < p_2$. Then by (5.6) and (5.7),

$$\begin{aligned} S(x, y; E, \Omega) &\leqslant xp_1^{-k} + \sum_{a=1}^{k} H(a, z_a) \\ &\leqslant xp_1^{-k} + \sum_{1 \leqslant a \leqslant v} H(a, z_a) + \sum_{v < a \leqslant p_1 v} H(a, z_a) \\ &+ \sum_{p_1 v < a \leqslant Q} H(a, z_a). \end{aligned} \tag{5.8}$$

For $1 \leqslant a \leqslant v$, take $z_a = 1$. With this choice, we have

$$\begin{aligned} \sum_{1 \leqslant a \leqslant v} H(a, z_a) &\ll xp_1^{-k} \sum_{1 \leqslant a \leqslant v} p_1^a \ll xp_1^{-y+v} \\ &\ll xp_1^{-y} e^{(p_1 - 1)v}. \end{aligned} \tag{5.9}$$

For $v < a \leqslant p_1 v$, the quantity $(z-1)v - a \log z$ in (5.7) is minimized by taking $z = a/v = z_a$. With this choice of z_a, we have $1 < z_a \leqslant p_1$ and

$$p_2 (p_2 - z_a)^{-1} \leqslant p_2 (p_2 - p_1)^{-1} \leqslant 1 + p_1,$$

so

$$H(a, z_a) \leqslant c_{39}(p_1) xp_1^{a-k} e^{(p_1 - 1)\Lambda} (v^a e^{-v} / a^a e^{-a}).$$

By Stirling's formula, $a^a e^{-a} \gg a!\, a^{-1/2}$, so we get

$$\begin{aligned} \sum_{v < a \leqslant p_1 v} H(a, z_a) &\leqslant c_{40}(p_1) xp_1^{-y} v^{1/2} e^{-v + p_1 \Lambda} \sum_{v < a \leqslant p_1 v} \frac{(p_1 v)^a}{a!} \\ &\leqslant c_{40}(p_1) xp_1^{-y} v^{1/2} e^{(p_1 - 1)v + p_1 \Lambda}. \end{aligned} \tag{5.10}$$

For $p_1 v < a \leqslant Q$, we let all the numbers z_a have the same value $p_1(1+\theta)$, where θ is a real number about which we assume only that $0 < \theta < p_2 p_1^{-1} - 1$ (the last inequality being needed in order to have $z_a < p_2$). With this choice of z_a, (5.7) yields

$$\sum_{p_1 v < a \leqslant Q} H(a, z_a)$$

$$\leqslant p_2 \{p_2 - p_1(1+\theta)\}^{-1} x p_1^{-k} \exp\{(p_1 - 1 + p_1\theta)(v+\Lambda) + 4p_1(1+\theta)\}$$

$$\times \sum_{p_1 v < a \leqslant Q} (1+\theta)^{-a}. \tag{5.11}$$

The last sum on the right does not exceed

$$\sum_{a > p_1 v} (1+\theta)^{-a} < (1+\theta)\,\theta^{-1}(1+\theta)^{-p_1 v}. \tag{5.12}$$

After combining this estimate with (5.11), we would like to minimize the contribution of the essential terms $e^{p_1 \theta v}\,\theta^{-1}(1+\theta)^{-p_1 v}$. Since

$$\log(1+\theta) \geqslant \theta - \theta^2/2 \qquad \text{for} \quad \theta \geqslant 0, \tag{5.13}$$

we have

$$p_1 \theta v - \log\theta - p_1 v \log(1+\theta) \leqslant -\log\theta + p_1 v \theta^2/2,$$

and here the right-hand side would be minimized by taking θ to be $(p_1 v)^{-1/2}$. However, we must also choose $\theta < p_2 p_1^{-1} - 1$ (so that $z_a < p_2$). If we take

$$\theta = (2p_1 v^{1/2})^{-1}, \tag{5.14}$$

then because of our assumption that $v \geqslant 1$, we have

$$\theta \leqslant (2p_1)^{-1} < p_2 p_1^{-1} - 1.$$

Combining (5.11), (5.12), (5.13), and (5.14), and observing that

$$p_2\{p_2 - p_1(1+\theta)\}^{-1} \leqslant p_2(p_2 - p_1 - 1/2)^{-1}$$
$$= 1 + (p_1 + 1/2)(p_2 - p_1 - 1/2)^{-1} \leqslant c_{41}(p_1),$$

we obtain finally

$$\sum_{p_1 v < a \leqslant Q} H(a, z_a) \leqslant c_{42}(p_1)\, x p_1^{-\nu} v^{1/2} e^{(p_1 - 1)v + p_1 \Lambda}. \tag{5.15}$$

The theorem now follows from (5.8), (5.9), (5.10), and (5.15). Q.E.D.

Since

$$E(x) \leqslant \sum_{p \leqslant x} p^{-1} = \log_2 x + O(1) \qquad \text{for} \quad x \geqslant 2,$$

one would always want to choose $v \leqslant \log_2 x$. Thus (1.23) is superior to (5.4) whenever $y \geqslant (\log_2 x)^{1/2}$. Furthermore, consideration of derivatives shows that

$$y - v - y \log(y/v) \leqslant (p_1 - 1)v - y \log p_1 \qquad \text{for} \qquad 0 < v \leqslant y \leqslant p_1 v,$$

and hence Lemma 5.3 is superior to Theorem 1.21 whenever

$$1 \leqslant v \leqslant y \leqslant p_1 v - v^{1/2}.$$

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