THE REPRESENTATION THEORY OF SL(2, R), A NON-INFINITESIMAL APPROACH

Autor(en): **Koornwinder, Tom H.**

Objekttyp: **Article**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **28 (1982)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **21.07.2024**

Persistenter Link: https://doi.org/10.5169/seals-52233

THE REPRESENTATION THEORY OF $SL(2, \mathbf{R})$, A NON-INFINITESIMAL APPROACH

by Tom H. KOORNWINDER

ABSTRACT

The representation theory of $SL(2, \mathbf{R})$ is developed by the use of non-infinitesimal methods. This approach is based on an explicit knowledge of the matrix elements of the principal series with respect to the K-basis. The irreducible subquotient representations of the principal series are determined, and also their Naimark equivalences and unitarizability. All irreducible K-unitary, K-finite representations of $SL(2, \mathbf{R})$ are classified, where an inversion formula for the generalized Abel transform provides an important tool.

1. INTRODUCTION

In 1947 two papers appeared on the representation theory of the two prototypes of noncompact semisimple Lie groups, namely by BARGMANN [2] on $SL(2, \mathbf{R})$ and by GELFAND & NAIMARK [18] on $SL(2, \mathbf{C})$. The methods in the two papers are surprisingly different. Bargmann uses the infinitesimal (i.e. Lie algebraic) approach, while Gelfand & Naimark prefer non-infinitesimal (global) methods. In subsequent work to generalize these results for arbitrary noncompact semisimple Lie groups, the Bargmann approach has proved to be most successful, in particular by the work of Harish-Chandra. (However, it is interesting to note MAUTNER's [31] review of HARISH-CHANDRA's paper [22].)

Without denying the success of the infinitesimal approach, I want to add some motivation for a paper which favours the global approach:

(a) *The didactic argument.* The global approach is a more natural and direct one and it does not require so much sophisticated functional analysis as the infinitesimal approach.

(b) *Spin off to the theory of special functions and related harmonic analysis.* The global approach requires explicit knowledge of canonical matrix elements of representations as special functions. This provides new group-theoretic interpretations of well-known special functions and it also yields new interesting special functions.

(c) *The philosophical argument.* The representation theory of semisimple Lie groups is one of the great topics in mathematics at the moment. It is good to have several distinct philosophies existing beside each other for the development of this theory, where each philosophy provides a different insight.

In this paper a global approach to the representation theory of $SL(2, \mathbf{R})$ is presented. It is based on an explicit knowledge of the matrix elements of the principal series representations with respect to a basis which behaves nicely under the action of a maximal compact subgroup K.

Our program consists of four parts:

(i) Determine all irreducible subquotient representations of the principal series representations of $SL(2, \mathbf{R})$.

(ii) Determine which equivalence do exist between the representations in (i).

(iii) Prove that each irreducible representation of $SL(2, \mathbf{R})$ is equivalent to some representation in (i).

(iv) Which of the representations in (i) are unitarizable?

We will not only consider unitary representations, but, more generally, strongly continuous representations on a Hilbert space which are K-unitary and K-finite (cf. §2.1). Accordingly, we need a more general (but still non-infinitesimal) notion of equivalence than the notion of unitary equivalence, namely Naimark equivalence (cf. §4.1).

The four parts of the above program will be treated in Sections 3, 4, 5 and 6, respectively. We start in Section 2 with the conputation of the canonical matrix elements of the principal series representations. They can be expressed in terms of hypergeometric or, more elegantly, Jacobi functions. These explicit expressions will be used throughout the paper. Each section ends with extensive bibliographic notes.

The theory required for parts (i), (ii) and (iv) of our program can be developed in the more general situation of Hilbert representations of a locally compact group G which are multiplicity free with respect to a compact subgroup K, cf. the author's report [27]. This would make the theory applicable to $SO_0(n, 1)$ and

$SU(n, 1)$. For convenience, in order to avoid matrix manipulations, we restrict ourselves here to the case that the compact subgroup K is abelian.

The results of this paper may be generalized rather easily to the universal covering group of $SL(2, \mathbf{R})$. The extension to $SL(2, \mathbf{C})$ was done by KOSTERS [28], see also NAIMARK [34, ch. 3, §9]. Hopefully, an extension to $SO_0(n, 1)$ and $SU(n, 1)$ is feasible.

The reader of this paper is supposed to already have a modest knowledge about certain elements of semisimple Lie theory, like principal series and spherical functions. Suitable references will be given. Some of this preliminary material can also be found in the earlier version [27]. Modern accounts of the infinitesimal approach to $SL(2, \mathbf{R})$ can be found, for instance, in SCHMID [36, §2] or VAN DIJK [9]. TAKAHASHI [42] also presented a global approach to $SL(2, \mathbf{R})$, partly based on an earlier version of the present paper, partly (the global proof of Theorem 5.4) independently.

Finally, I would like to thank G. van Dijk and M. Flensted-Jensen for useful comments.

2. THE CANONICAL MATRIX ELEMENTS OF THE PRINCIPAL SERIES

2.1. PRELIMINARIES

Let G be a locally compact group satisfying the second axiom of countability (lcsc. group). A *Hilbert representation* of G is a strongly continuous but not necessarily unitary representation τ of G on some Hilbert space $\mathscr{H}(\tau)$ (which is always assumed to be separable). Let K be a compact subgroup of G. A Hilbert representation τ of G is called *K-unitary* if the restriction $\tau|_K$ of τ to K is a unitary representation of K. A Hilbert representation τ of G is called *K-finite* respectively *K-multiplicity free* if τ is K-unitary and each $\delta \in \hat{K}$ has finite multiplicity respectively multiplicity 1 or 0 in $\tau|_K$. If τ is K-multiplicity free then the *K-content* $\mathscr{M}(\tau)$ of τ is the set of all $\delta \in \hat{K}$ which have multiplicity 1 in $\tau|_K$.

Let K be a compact abelian subgroup of G and let τ be a K-multiplicity free representation of G. Choose an orthogonal basis $\{\phi_\delta \mid \delta \in \mathscr{M}(\tau)\}$ of $\mathscr{H}(\tau)$ such that

$$\tau(k)\phi_\delta = \delta(k)\phi_\delta, \qquad \delta \in \mathscr{M}(\tau), k \in K.$$

We call $\{\phi_\delta\}$ a *K-basis* for $\mathscr{H}(\tau)$ and the functions $\tau_{\gamma\delta}(\gamma, \delta \in \mathscr{M}(\tau))$, defined by

$$\tau_{\gamma,\delta}(g) := (\tau(g)\phi_\delta, \phi_\gamma), \quad g \in G, \tag{2.1}$$

the *canonical matrix elements* of τ (with respect to K).

2.2. The principal series

Let G be a connected noncompact real semisimple Lie group with finite center. Let $G = KAN$ be an Iwasawa decomposition. For $g \in G$ write $g = u(g)\exp(H(g))n(g)$, where $u(g) \in K$, $H(g) \in \mathfrak{a}$ (the Lie algebra of A) and $n(g) \in N$. Let $\rho \in \mathfrak{a}^*$ be half the sum of the positive roots. Let M be the centralizer of A in K. For $\xi \in \hat{M}$, $\lambda \in \mathfrak{a}_{\mathbf{C}}^*$ the *principal series* representation $\pi_{\xi,\lambda}$ of G is obtained by inducing the (not necessarily unitary) finite-dimensional irreducible representation $man \to e^{\lambda(\log a)}\xi(m)$ of the subgroup MAN. In the so-called compact picture we have the following realization of $\pi_{\xi,\lambda}$ (cf. Wallach [45, §8.3]):

$$(2.2)\qquad \big(\pi_{\xi,\lambda}(g)f\big)(k) = e^{-(\rho+\lambda)(H(g^{-1}k))}\, f\big(u(g^{-1}k)\big)\,, \quad f \in L^2_\xi\big(K, \mathcal{H}(\xi)\big)\,, \quad k \in K, g \in G\,.$$

Here the Hilbert space $L^2_\xi(K, \mathcal{H}(\xi))$ consists of all $\mathcal{H}(\xi)$-valued L^2-functions f on K such that $f(km) = \xi(m^{-1})f(k)$, $k \in K$, $m \in M$. The representation $\pi_{\xi,\lambda}$ is a K-unitary Hilbert representation. It is unitary if $\lambda \in ia^*$. By Frobenius reciprocity, $\pi_{\xi,\lambda}$ is K-finite and $\pi_{\xi,\lambda}$ is K-multiplicity free if each $\delta \in \hat{K}$ is M-multiplicity free.

Let us now specialize the above results to $G = SL(2, \mathbf{R})$. It is convenient to work with the group $G = SU(1, 1)$, isomorphic to $SL(2, \mathbf{R})$:

$$(2.3)\qquad G := \left\{ g_{\alpha,\beta} = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix};\quad \alpha, \beta \in \mathbf{C}, |\alpha|^2 - |\beta|^2 = 1 \right\}.$$

Let

$$(2.4)\qquad K := \left\{ u_\theta = \begin{pmatrix} e^{\frac{1}{2}i\theta} & 0 \\ 0 & e^{-\frac{1}{2}i\theta} \end{pmatrix};\quad 0 \leqslant \theta < 4\pi \right\},$$

$$(2.5)\qquad A := \left\{ a_t = \begin{pmatrix} ch\frac{1}{2}t & sh\frac{1}{2}t \\ sh\frac{1}{2}t & ch\frac{1}{2}t \end{pmatrix} = \exp t \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}; t \in \mathbf{R} \right\},$$

$$(2.6)\qquad N := \left\{ n_z = \begin{pmatrix} 1+\frac{1}{2}iz & -\frac{1}{2}iz \\ \frac{1}{2}iz & 1-\frac{1}{2}iz \end{pmatrix}; z \in \mathbf{R} \right\}.$$

Then $G = KAN$ is an Iwasawa decomposition for $G = SU(1, 1)$, $\rho(\log a_t) = \frac{1}{2}t$ and $M = \{u_0, u_{2\pi}\}$. $\hat{M}$ consists of the two one-dimensional representations

(2.7) $$u_\theta \to e^{i\xi\theta}, \quad u_\theta \in M, \xi = 0 \text{ or } \tfrac{1}{2}.$$

Let $L^2_\xi(K)$ consist of all $f \in L^2(K)$ such that $f(u_{\psi+2\pi}) = f(u_\psi)$ or $-f(u_\psi)$ according to whether $\xi = 0$ or $\frac{1}{2}$, respectively.

Now, by using explicit expressions for the factors in the Iwasawa decomposition of $g^{-1}_{\alpha,\beta} u_\psi$ (cf. TAKAHASHI [39, §1]) we can write (2.2) in the case G $SU(1, 1)$ as follows:

(2.8) $$(\pi_{\xi,\lambda}(g_{\alpha,\beta})f)(u_\psi) := |\bar{\alpha} e^{\frac{1}{2}i\psi} - \beta e^{-\frac{1}{2}i\psi}|^{-2\lambda-1} f(u_{\psi'}),$$
$$\psi' := 2\arg(\bar{\alpha} e^{\frac{1}{2}i\psi} - \beta e^{-\frac{1}{2}i\psi}), \quad g_{\alpha,\beta} \in G, u_\psi \in K, f \in L^2_\xi(K),$$
$$\xi = 0 \text{ or } \tfrac{1}{2}, \lambda \in \mathbf{C}.$$

On putting $g_{\alpha,\beta} := u_\theta \in K$ we get

(2.9) $$(\pi_{\xi,\lambda}(u_\theta)f)(u_\psi) = f(u_{\psi-\theta}), \quad f \in L^2_\xi(K), u_\theta, u_\psi \in K,$$

which again shows that $\pi_{\xi,\lambda}$ is K-unitary. $\hat{K}$ consists of the representations

(2.10) $$\delta_n(u_\theta) := e^{in\theta}, \quad u_\theta \in K,$$

where n runs through the set $\frac{1}{2}\mathbf{Z}$, i.e., $2n \in \mathbf{Z}$. An orthogonal basis for $L^2_\xi(K)$ is given by the functions

(2.11) $$\phi_n(u_\psi) := e^{-in\psi}, \quad u_\psi \in K,$$

where n runs through the set $\mathbf{Z} + \xi := \{m+\xi \mid m \in \mathbf{Z}\}$. Then

(2.12) $$\pi_{\xi,\lambda}(u_\theta)\phi_n = \delta_n(u_\theta)\phi_n, \quad u_\theta \in K, n \in \mathbf{Z} + \xi.$$

Thus $\pi_{\xi,\lambda}$ is K-multiplicity free,

(2.13) $$\mathcal{M}(\pi_{\xi,\lambda}) = \{\delta_n \in \hat{K} \mid n \in \mathbf{Z} + \xi\},$$

the ϕ_n's form a K-basis for $L^2_\xi(K)$ and the canonical matrix elements of $\pi_{\xi,\lambda}$ are

(2.14) $$\pi_{\xi,\lambda,m,n}(g) = (\pi_{\xi,\lambda}(g)\phi_n, \phi_m), \quad g \in G, m, n \in \mathbf{Z} + \xi.$$

Because of the Cartan decomposition $G = KAK$, $\pi_{\xi,\lambda,m,n}$ is completely determined by its restriction to A. It follows from (2.8) and (2.11) that

$$(\pi_{\xi,\lambda}(a_t)\phi_n)(u_\psi) = |ch\tfrac{1}{2}t\, e^{\frac{1}{2}i\psi} - sh\tfrac{1}{2}t\, e^{-\frac{1}{2}i\psi}|^{-2\lambda+2n-1}$$
$$\cdot (ch\tfrac{1}{2}t\, e^{\frac{1}{2}i\psi} - sh\tfrac{1}{2}t\, e^{-\frac{1}{2}i\psi})^{-2n}.$$

Hence

$$\pi_{\xi,\lambda,m,n}(a_t) = (ch\tfrac{1}{2}t)^{-2\lambda-1} \tag{2.15}$$

$$\cdot \frac{1}{4\pi}\int_0^{4\pi} (1-th\tfrac{1}{2}t\, e^{i\psi})^{-\lambda+n-1/2}(1-th\tfrac{1}{2}t\, e^{-i\psi})^{-\lambda-n-1/2} e^{i(m-n)\psi} d\psi\,,$$

$$t \in \mathbf{R},\ m,n \in \mathbf{Z} + \xi\,.$$

The following symmatry is evident from (2.15):

$$\pi_{\xi,\lambda,-m,-n}(a_t) = \pi_{\xi,\lambda,m,n}(a_t)\,. \tag{2.16}$$

2.3. Calculation of the canonical matrix elements

Let us calculate the integral (2.15). In view of (2.16) we can suppose $m \geqslant n$. The binomial expansion

$$(1-z)^{-a} = \sum_{k=0}^{\infty} \frac{(a)_k}{k!} z^k, \quad |z| < 1, a \in \mathbf{C}\,, \tag{2.17}$$

where

$$(a)_k := a(a+1)\dots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}\,, \tag{2.18}$$

can be substituted for the first two factors in the integrand of (2.15). Now interchange the order of summation and integration and perform the integration in each term. Then we obtain $(m \geqslant n)$

$$\pi_{\xi,\lambda,m,n}(a_t) = \frac{(\lambda+n+\frac{1}{2})_{m-n}}{(m-n)!}(sh\tfrac{1}{2}t)^{m-n}(ch\tfrac{1}{2}t)^{n-m-2\lambda-1} \tag{2.19}$$

$$\cdot\, {}_2F_1(\lambda+m+\tfrac{1}{2}, \lambda-n+\tfrac{1}{2}; m-n+1; (th\tfrac{1}{2}t)^2)\,,$$

where the ${}_2F_1$ denotes a *hypergeometric series*, defined by

$${}_2F_1(a, b; c; z) := \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k\, k!} z^k, \quad |z| < 1, a,b,c \in \mathbf{C}\,, \tag{2.20}$$

cf. [10, Vol. I, Ch. 2].

The expression (2.20) is clearly symmetric in a and b. As a function of z, the ${}_2F_1$ has an analytic continuation to a one-valued function on $\mathbf{C}\backslash[1, \infty)$. Application of the transformation formulas

$$(2.21)\qquad {}_2F_1(a,b;c;z) = (1-z)^{-b}{}_2F_1\left(c-a,b;c;\frac{z}{z-1}\right) = (1-z)^{-a}{}_2F_1\left(a,c-b;c;\frac{z}{z-1}\right)$$

(cf. [10, Vol. I, §2.1 (22)]) to (2.19) yields ($m \geq n$):

$$(2.22)\qquad \pi_{\xi,\lambda,m,n}(a_t)$$
$$= \frac{(\lambda+n+\frac{1}{2})_{m-n}}{(m-n)!}(sh\tfrac{1}{2}t)^{m-n}(ch\tfrac{1}{2}t)^{-m-n}{}_2F_1(\lambda-n+\tfrac{1}{2}, -\lambda-n+\tfrac{1}{2}; m-n+1; -(sh\tfrac{1}{2}t)^2)$$
$$= \frac{(\lambda+n+\frac{1}{2})_{m-n}}{(m-n)!}(sh\tfrac{1}{2}t)^{m-n}(ch\tfrac{1}{2}t)^{m+n}{}_2F_1(\lambda+m+\tfrac{1}{2}, -\lambda+m+\tfrac{1}{2}; m-n+1; -(sh\tfrac{1}{2}t)^2).$$

It is more elegant to express the hypergeometric functions in (2.22) in terms of *Jacobi functions* $\phi_\mu^{(\alpha,\beta)}$ ($\mu,\alpha,\beta \in \mathbf{C}$, $\alpha \notin \{-1,-2,...\}$), which are defined on $\mathbf{R}$ by

$$(2.23)\qquad \phi_\mu^{(\alpha,\beta)}(t) := {}_2F_1(\tfrac{1}{2}(\alpha+\beta+1+i\mu), \tfrac{1}{2}(\alpha+\beta+1-i\mu); \alpha+1; -(sht)^2)$$

(cf. Koornwinder [36, §2]). Clearly,

$$(2.24)\qquad \phi_\mu^{(\alpha,\beta)}(0) = 1,$$

$$(2.25)\qquad \phi_\mu^{(\alpha,\beta)}(t) = \phi_\mu^{(\alpha,\beta)}(-t),$$

$$(2.26)\qquad \phi_\mu^{(\alpha,\beta)}(t) = \phi_{-\mu}^{(\alpha,\beta)}(t).$$

The function $\phi_\mu^{(\alpha,\beta)}$ satisfies the differential equation

$$(2.27)\qquad (\Delta_{\alpha,\beta}(t))^{-1}\frac{d}{dt}\left(\Delta_{\alpha,\beta}(t)\frac{du(t)}{dt}\right) = -(\mu^2+(\alpha+\beta+1)^2)u(t),$$

where

$$\Delta_{\alpha,\beta}(t) := (sht)^{2\alpha+1}(cht)^{2\beta+1},$$

and $u := \phi_\mu^{\alpha,\beta}$ is the unique solution of (2.27) which is regular at $t = 0$ and satisfies $u(0) = 1$. For fixed $\alpha > -1$, $\beta \in \mathbf{R}$, Jacobi functions $\phi_\mu^{(\alpha,\beta)}$ form a continuous orthogonal system with respect to the measure $\Delta_{\alpha,\beta}(t)dt$, $t > 0$.

Substitution of (2.23) and (2.22) yields $(m \geqslant n)$:

(2.28)
$$\pi_{\xi,\lambda,m,n}(a_t)$$
$$= \frac{(\lambda+n+\frac{1}{2})_{m-n}}{(m-n)!}(sh\tfrac{1}{2}t)^{m-n}(ch\tfrac{1}{2}t)^{-m-n}\phi_{2i\lambda}^{(m-n,\,-m-n)}(\tfrac{1}{2}t)$$
$$= \frac{(\lambda+n+\frac{1}{2})_{m-n}}{(m-n)!}(sh\tfrac{1}{2}t)^{m-n}(ch\tfrac{1}{2}t)^{m+n}\phi_{2i\lambda}^{(m-n,\,m+n)}(\tfrac{1}{2}t)\,.$$

Application of (2.16) gives a similar result in the case $m < n$. Finally we conclude:

THEOREM 2.1. *The canonical matrix elements* $\pi_{\xi,\lambda,m,n}(a_t)$ $(\lambda \in \mathbf{C};\ \xi = 0$ or $\frac{1}{2};\ m, n \in \mathbf{Z} + \xi;\ t \in \mathbf{R})$ *of* $SU(1, 1)$ *can be expressed in terms of Jacobi functions by*

(2.29)
$$\pi_{\xi,\lambda,m,n}(a_t) = \frac{c_{\xi,\lambda,m,n}}{(|m-n|)!}(sh\tfrac{1}{2}t)^{|m-n|}(ch\tfrac{1}{2}t)^{m+n}\phi_{2i\lambda}^{(|m-n|,\,m+n)}(\tfrac{1}{2}t)\,,$$

where

(2.30)
$$c_{\xi,\lambda,m,n} := \begin{cases}(\lambda+n+\frac{1}{2})_{m-n} & \text{if } m \geqslant n\,,\\ (\lambda-n+\frac{1}{2})_{n-m} & \text{if } n \geqslant m\,.\end{cases}$$

In view of (2.24), formulas (2.29) and (2.30) describe the asymptotics of $\pi_{\xi,\lambda,m,n}$ near $t = 0$.

2.4. NOTES

2.4.1. The principal series of representations was first written down for $SL(2, \mathbf{R})$ by BARGMANN [2], for $SL(2, \mathbf{C})$ by GELFAND & NAIMARK [18], and for a general noncompact semisimple Lie group by HARISH-CHANDRA [21, §12].

2.4.2. BARGMANN [2, §10] already obtained explicit expressions in terms of hypergeometric functions for the canonical matrix elements of the irreducible unitary representations of $SL(2, \mathbf{R})$. He solved the differential equation satisfied by these matrix elements, which is obtained from the Casimir operator. VILENKIN [43, Ch. VI, §3] gives a derivation of these expressions which is similar to our derivation in §2.4, starting from the integral representation (2.15).

2.4.3. It follows from the present paper that the spherical functions for $SL(2, \mathbf{R})$ can be expressed as Jacobi functions of order $(\alpha, \beta) = (0, 0)$. More generally, the spherical functions on any noncompact real semisimple Lie group of rank 1 (i.e., $\dim(A) = 1$) can be written as Jacobi functions of certain order (cf. HARISH-CHANDRA [23, §13]). This motivated FLENSTED-JENSEN [14] to study harmonic analysis for Jacobi function expansions of quite general order (α, β), $\alpha \geqslant \beta \geqslant -\frac{1}{2}$. This research was continued in several papers by Flensted-Jensen and the author.

3. THE IRREDUCIBLE SUBQUOTIENT REPRESENTATIONS OF THE PRINCIPAL SERIES

3.1. SUBQUOTIENT REPRESENTATIONS

We start with the definition and some general properties and next derive an irreducibility criterium (Theorem 3.2) and a decomposition theorem 3.3.

Let G be a lcsc. group and let τ be a Hilbert representation of G. Let $\mathscr{H}_0$ be a closed subspace of $\mathscr{H}(\tau)$ and let P_0 be the orthogonal projection from $\mathscr{H}(\tau)$ onto $\mathscr{H}_0$. Define

$$\tau_0(g)v := P_0\tau(g)v\,, \quad g \in G, v \in \mathscr{H}_0\,. \tag{3.1}$$

Then $\tau(g) \in \mathscr{L}(\mathscr{H}_0)$ for each $g \in G$, $\tau_0(e) = id.$, and $g \to \tau_0(g)v: G \to \mathscr{H}_0$ is continuous for each $v \in \mathscr{H}_0$. If also

$$\tau_0(g_1 g_2) = \tau_0(g_1)\tau_0(g_2)\,, \quad g_1, g_2 \in G\,, \tag{3.2}$$

then τ_0 is a Hilbert representation of G on $\mathscr{H}_0$ and it is called a *subquotient representation* of τ. Formula (3.2) is clearly valid if $\mathscr{H}_0$ is an *invariant subspace* of $\mathscr{H}(\tau)$, i.e., if $\tau(g)v \in \mathscr{H}_0$ for all $g \in G$, $v \in \mathscr{H}_0$. In that case, τ_0 is called a *subrepresentation* of τ.

LEMMA 3.1. *Let* $\mathscr{H}_0$ *be a closed subspace of* $\mathscr{H}(\tau)$, *let* $\mathscr{H}_2$ *be the closed G-invariant subspace of* $\mathscr{H}(\tau)$ *which is generated by* $\mathscr{H}_0$ *and let* $\mathscr{H}_1 := \mathscr{H}_2 \cap \mathscr{H}_0^{\perp}$. *Then* τ_0 *is a subquotient representation if and only if* $\mathscr{H}_1$ *is G-invariant.*

Proof. Let P_0 and P_1 denote the orthogonal projections on $\mathscr{H}_0$ and $\mathscr{H}_1$, respectively. It follows from (3.1) that

$$\begin{aligned}&\tau_0(g_1g_2)v - \tau_0(g_1)\tau_0(g_2)v\\ &= P_0\tau(g_1)P_1\tau(g_2)v\,, \quad g_1, g_2 \in G, v \in \mathscr{H}_0\,.\end{aligned}$$

$\mathscr{H}_1$ is the closed linear span of all elements $P_1\tau(g_2)v$, $g_2 \in G$, $v \in \mathscr{H}_0$. So (3.2) holds iff $P_0\tau(g_1)w = 0$ for all $g_1 \in G$, $w \in \mathscr{H}_1$. □

Let K be a compact subgroup of G and suppose that τ is K-unitary. Let τ_0 be a subquotient representation of τ on $\mathscr{H}_0$ and let $\mathscr{H}_1$ and $\mathscr{H}_2$ be as in Lemma 3.1. Then $\mathscr{H}_2$ and $\mathscr{H}_1$ are G-invariant subspaces, so $\mathscr{H}_0 = \mathscr{H}_2 \cap \mathscr{H}_1^\perp$ is K-invariant. It follows that τ_0 is K-unitary and that $\tau_0(k)v = \tau(k)v$, $k \in K$, $v \in \mathscr{H}_0$. If K is compact abelian and if τ is K-multiplicity free then τ_0 is also K-multiplicity free, $\mathscr{M}(\tau_0) \subset \mathscr{M}(\tau)$ and $\tau_{0,\gamma,\delta}(g) = \tau_{\gamma,\delta}(g)$ for $\gamma, \delta \in \mathscr{M}(\tau_0)$, $g \in G$.

Let again K be a compact abelian subgroup of G and τ a K-multiplicity free Hilbert representation of G. Let $\mathscr{H}_0$ be a K-invariant closed subspace of $\mathscr{H}(\tau)$. Then, by Lemma 3.1, τ_0 defined by (3.1) is a subquotient representation if and only if we can partition the K-basis for $\mathscr{H}(\tau)$ into three parts, the first part providing a basis for $\mathscr{H}_0$, such that, for each $g \in G$, the corresponding 3×3 block matrix of $(\tau_{\gamma\delta}(g))$ takes the form

$$\begin{pmatrix} * & 0 & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}. \tag{3.3}$$

THEOREM 3.2. *Let K be a compact abelian subgroup of the lcsc. group G and let τ be a K-multiplicity free Hilbert representation of G. Let τ_0 be a subquotient representation of τ. Then the following three statements are equivalent:*

(a) τ_0 *is irreducible.*

(b) *For some* $\delta \in \mathscr{M}(\tau_0)$ *we have* $\tau_{\gamma\delta} \neq 0 \neq \tau_{\delta\gamma}$ *for all* $\gamma \in \mathscr{M}(\tau_0)$.

(c) *For all* $\gamma, \delta \in \mathscr{M}(\tau_0)$ *we have* $\tau_{\gamma\delta} \neq 0$.

Proof. First note: if $v \in \mathscr{H}(\tau_0)$ and $(v, \phi_\gamma) \neq 0$ for some $\gamma \in \mathscr{M}(\tau_0)$ then ϕ_γ (element of the K-basis) belongs to the τ_0-invariant subspace of $\mathscr{H}(\tau_0)$ generated by v. Indeed,

$$(v, \phi_\gamma)\phi_\gamma = \int_K \gamma(k^{-1})\tau(k)v\, dv$$

and

$$\tau(k)v = \tau_0(k)v\,.$$

(b) $\Rightarrow$ (a): Let $0 \neq v \in \mathscr{H}(\tau_0)$. Let $\mathscr{H}_1$ be the τ_0-invariant subspace of $\mathscr{H}(\tau_0)$ generated by v. Then $\phi_\gamma \in \mathscr{H}_1$ for some $\gamma \in \mathscr{M}(\tau_0)$. Now, for some $g \in G$,

$$(\tau_0(g)\phi_\gamma, \phi_\delta) = \tau_{0,\delta,\gamma}(g) = \tau_{\delta,\gamma}(g) \neq 0\,,$$

so $\tau_0(g)\phi_\gamma$ and ϕ_δ are in $\mathscr{H}_1$. For each $\beta \in \mathscr{M}(\tau_0)$ we have $(\tau_0(g)\phi_\delta, \phi_\beta) = \tau_{\beta\delta}(g) \neq 0$ for some $g \in G$. Thus $\phi_\beta \in \mathscr{H}_1$ for all $\beta \in \mathscr{M}(\tau_0)$, so $\mathscr{H}_1 = \mathscr{H}(\tau_0)$.

(a) $\Rightarrow$ (c): Suppose $\tau_{\gamma\delta} = 0$ for some $\gamma, \delta \in \mathscr{M}(\tau_0)$. Then, for all $g \in G$, $(\tau_0(g)\phi_\delta, \phi_\gamma) = 0$. Hence, the τ_0-invariant subspace of $\mathscr{H}(\tau_0)$ generated by ϕ_δ is orthogonal to ϕ_γ, so τ_0 is not irreducible.

(c) $\Rightarrow$ (b): Clear. $\square$

Let τ be K-multiplicity free, K being compact abelian. Define a relation $\prec$ on $\mathscr{M}(\tau)$ by: $\gamma \prec \delta$ iff $\tau_{\gamma,\delta} \neq 0$. Then $\gamma \prec \delta$ iff ϕ_γ is in the τ-invariant subspace of $\mathscr{H}(\tau)$ generated by ϕ_δ. It follows that

$$\beta \prec \gamma \text{ and } \gamma \prec \delta \Rightarrow \beta \prec \delta$$

Define a relation $\sim$ on $\mathscr{M}(\tau)$ by: $\gamma \sim \delta$ iff $\tau_{\gamma,\delta} \neq 0 \neq \tau_{\delta,\gamma}$. It follows that $\sim$ is an equivalence relation on $\mathscr{M}(\tau)$ and that, if $\tau_{\gamma,\delta} \neq 0, \alpha \sim \gamma, \beta \sim \delta$ then $\tau_{\alpha,\beta} \neq 0$. It follows that, for a given equivalence set, we can partition $\mathscr{M}(\tau)$ into three parts, the first part being the equivalence set, such that the corresponding 3×3 block matrix for $(\tau_{\gamma\delta}(g))$ takes the form (3.3). In view of Theorem 3.2 this proves:

THEOREM 3.3. *Let G be a lcsc. group with compact abelian subgroup K and let τ be a K-multiplicity free representation of G. Then there is a unique orthogonal decomposition of $\mathscr{H}(\tau)$ into subspaces $\mathscr{H}(\tau_i)$, where the τ_i's are precisely the irreducible subquotient representations of τ.*

3.2. THE CASE $SU(1, 1)$

For $\lambda \in \mathbf{C}$, $\xi = 0$ or $\frac{1}{2}$, the representation $\pi_{\xi,\lambda}$ of $G = SU(1,1)$ on $L^2_\xi(K)$ (cf. (2.8)) is K-multiplicity free with K-content given by (2.13). By inspecting (2.29) for small but nonzero t and by using (2.24) it follows that

$$\pi_{\xi,\lambda,m,n} \neq 0 \Leftrightarrow \pi_{\xi,\lambda,m,n}|_A \neq 0 \Leftrightarrow c_{\xi,\lambda,m,n} \neq 0\,, \tag{3.4}$$

where $c_{\xi,\lambda,m,n}$ is given by (2.30). Combination of (3.4) with Theorems 3.2 and 3.3 yields:

THEOREM 3.4. *Depending on* ξ *and* λ, *the representation* $\pi_{\xi,\lambda}$ *of* $SU(1, 1)$ *has the following irreducible subquotient representations:*

(a) $\underline{\lambda + \xi \notin \mathbf{Z} + \frac{1}{2}:}$

$\pi_{\xi,\lambda}$ *is irreducible itself.*

(b) $\underline{\lambda = 0, \xi = \frac{1}{2}:}$

$\pi^+_{1/2,0}$ *on* Cl Span $\{\phi_{1/2}, \phi_{3/2}, \dots\}$,
$\pi^-_{1/2,0}$ *on* Cl Span $\{\dots, \phi_{-3/2}, \phi_{-1/2}\}$.

These are also subrepresentations.

(c) $\underline{\lambda + \xi \in \mathbf{Z} + \frac{1}{2}, \lambda > 0:}$

$\pi^+_{\xi,\lambda}$ *on* Cl Span $\{\phi_{\lambda+1/2}, \phi_{\lambda+3/2}, \dots\}$,
$\pi^-_{\xi,\lambda}$ *on* Cl Span $\{\dots, \phi_{-\lambda-3/2}, \phi_{-\lambda-1/2}\}$,
$\pi^0_{\xi,\lambda}$ *on* Span $\{\phi_{-\lambda+1/2}, \phi_{-\lambda+3/2}, \dots, \phi_{\lambda-1/2}\}$.

Among these $\pi^+_{\xi,\lambda}$ *and* $\pi^-_{\xi,\lambda}$ *are subrepresentations.*

(d) $\underline{\lambda + \xi \in \mathbf{Z} + \frac{1}{2}, \lambda < 0:}$

$\pi^+_{\xi,\lambda}$ *on* Cl Span $\{\phi_{-\lambda+1/2}, \phi_{-\lambda+3/2}, \dots\}$,
$\pi^-_{\xi,\lambda}$ *on* Cl Span $\{\dots, \phi_{\lambda-3/2}, \phi_{\lambda-1/2}\}$,
$\pi_{\xi,\lambda}$ *on* Span $\{\phi_{\lambda+1/2}, \phi_{\lambda+3/2}, \dots, \phi_{-\lambda-1/2}\}$.

Among these $\pi^0_{\xi,\lambda}$ *is a subrepresentation.*

Proof.

(a) $c_{\xi,\lambda,m,n} \neq 0$.
(b) $c_{1/2,0,m,n} \neq 0 \Leftrightarrow m, n \leqslant -\frac{1}{2}$ or $m, n \geqslant \frac{1}{2}$.
(c) $c_{\xi,\lambda,m,n} \neq 0 \Leftrightarrow -\lambda + \frac{1}{2} \leqslant n \leqslant \lambda - \frac{1}{2}$ or $m, n \leqslant -\lambda - \frac{1}{2}$ or $m, n \geqslant \lambda + \frac{1}{2}$.

Thus $c_{\xi, \lambda, m, n}$ has block matrix

$$\begin{array}{l c c c}
 & n \leqslant -\lambda - \frac{1}{2} & -\lambda + \frac{1}{2} \leqslant n \leqslant \lambda - \frac{1}{2} & n \geqslant \lambda + \frac{1}{2} \\
\begin{array}{l} m \leqslant -\lambda - \frac{1}{2} \\ -\lambda + \frac{1}{2} \leqslant m \leqslant \lambda - \frac{1}{2} \\ m \geqslant \lambda + \frac{1}{2} \end{array} & \multicolumn{3}{c}{\begin{pmatrix} * & \quad * \quad & 0 \\ 0 & * & 0 \\ 0 & * & * \end{pmatrix}}
\end{array}$$

where each starred block has all entries nonzero.

(d) $c_{\xi, \lambda, m, n} \neq 0 \Leftrightarrow \lambda + \frac{1}{2} \leqslant m \leqslant -\lambda - \frac{1}{2}$ or $m, n \leqslant \lambda - \frac{1}{2}$ or $m, n > -\lambda + \frac{1}{2}$. □

The finite-dimensional representation occurring in the above classification are the representations $\pi^0_{\xi, \lambda}(\lambda + \xi \in \mathbf{Z} + \frac{1}{2}, \lambda \neq 0)$.

3.3. Notes

3.3.1. In the case of the unitary principal series (λ imaginary), Theorem 3.4 was first proved by Bargmann [2, sections 6 and 7]. See van Dijk [9, Theorem 4.1] for the statement and (infinitesimal) proof of our Theorem 3.4 in the general case. A proof of Theorem 3.4 similar to our proof was earlier given by Barut & Phillips [3, §II (4)].

3.3.2. Theorem 3.4 in the case of imaginary and nonzero λ is contained in a general theorem by Bruhat [5, Theorem 7; 2]: For $\xi \in \hat{M}$, $\lambda \in i\mathfrak{a}$, the principal series representation $\pi_{\xi, \lambda}$ of G (cf. (2.2)) is irreducible if $s \,.\, \lambda \neq \lambda$ for all $s \neq e$ in the Weyl group for (G, K).

3.3.3. Gelfand & Naimark [18, §5.4, Theorem 1] proved the irreducibility of the unitary principal series for $SL(2, \mathbf{C})$ by a global method different from ours, working in a noncompact realization and calculating the "matrix elements" of the representation with respect to a (continuous) $\overline{N}$-basis.

3.3.4. Analogues of Theorems 3.2 and 3.3 can be formulated in the case of non-abelian K, cf. [27, Theorem 3.3]. In that case the canonical matrix elements $\tau_{\gamma, \delta}$ are matrix-valued functions. By using this method, Naimark [34, Ch. 3, §9, No. 15] examined the irreducibility of the nonunitary principal series for $SL(2, \mathbf{C})$, see also Kosters [28].

3.3.5. Further applications of the irreducibility criterium in Theorem 3.2 can be found in MILLER [32, Lemmas 3.2 and 4.5] for the Euclidean motion group of $\mathbf{R}^2$ and for the harmonic oscillator group, TAKAHASHI [39, §3.4] for the discrete series of $SL(2, \mathbf{R})$ and [41, p. 560, Cor. 2] for the spherical principal series of $F_{4(-20)}$.

3.3.6. The method of this section does not show in an *a priori* way that a K-multiplicity free principal series representation has only finitely many irreducible subquotient representations. Actually, this property holds quite generally, cf. WALLACH [45, Theorem 8.13.3].

4. EQUIVALENCES BETWEEN IRREDUCIBLE SUBQUOTIENT REPRESENTATIONS OF THE PRINCIPAL SERIES

4.1. NAIMARK EQUIVALENCE

In this subsection we derive a criterium (Theorem 4.5) for Naimark equivalence of K-multiplicity free representations. Lemmas 4.3 and 4.4 are preparations for its proof.

Let G be an lcsc. group.

Definition 4.1. Let σ and τ be Hilbert representations of G. The representation σ is called *Naimark related* to τ if there is a closed (possibly) unbounded) injective linear operator A from $\mathscr{H}(\sigma)$ to $\mathscr{H}(\tau)$ with domain $\mathscr{D}(A)$ dense in $\mathscr{H}(\sigma)$ and range $\mathscr{R}(A)$ dense in $\mathscr{H}(\tau)$ such that $\mathscr{D}(A)$ is σ-invariant and $A\sigma(g)v = \tau(G)Av$ for all $v \in \mathscr{D}(A)$, $g \in G$. Then we use the notation $\sigma \simeq \tau$ or $\sigma \underset{A}{\simeq} \tau$.

Naimark relatedness is not necessarily a transitive relation (cf. WARNER [48, p. 242]). However, we will see that it becomes an equivalence relation (called *Naimark equivalence*) when restricted to the class of unitary representations or of K-multiplicity free representations, K abelian.

Two unitary representations σ and τ of G are called *unitarily equivalent* if there is an isometry A from $\mathscr{H}(\sigma)$ onto $\mathscr{H}(\tau)$ such that $A\sigma(g)v = \tau(g)Av$ for all $v \in \mathscr{H}(\sigma)$, $g \in G$. Clearly unitary equivalence is an equivalence relation.

PROPOSITION 4.2. *Two unitary representations of an lcsc. group G are Naimark related if and only if they are unitarily equivalent.*

See WARNER [48, Prop. 4.3.1.4] for the proof.

Let K be a compact abelian subgroup of G. Let σ and τ be K-multiplicity free representations of G. Let $\{\phi_\delta\}$ and $\{\psi_\delta\}$ be K-bases for $\mathscr{H}(\sigma)$ and $\mathscr{H}(\tau)$, respectively.

LEMMA 4.3. *If* $\sigma \overset{A}{\simeq} \tau$ *then* $\mathscr{M}(\sigma) = \mathscr{M}(\tau)$, $\phi_\delta \in \mathscr{D}(A)$ *and* $\psi_\delta \in \mathscr{R}(A)$ $(\delta \in \mathscr{M}(\sigma))$, *and there are nonzero complex numbers* $c_\delta(\delta \in \mathscr{M}(\sigma))$ *such that*

$$(4.1) \qquad (Av, \psi_\delta) = c_\delta(v, \phi_\delta), \qquad v \in \mathscr{D}(A).$$

In particular

$$(4.2) \qquad A\phi_\delta = c_\delta \psi_\delta .$$

Proof. Let $\delta \in \mathscr{M}(\sigma)$. Let $v \in \mathscr{D}(A)$. We have, by the intertwining property of A,

$$\int_K \delta(k^{-1})\sigma(k)v\,dk = (v, \phi_\delta)\phi_\delta ,$$

$$\int_K \delta(k^{-1})A\sigma(k)v\,dk = \int_K \delta(k^{-1})\sigma(k)Av\,dk$$

$$= \begin{cases} (Av, \psi_\delta)\psi_\delta & \text{if } \delta \in \mathscr{M}(\tau), \\ 0 & \text{if } \delta \notin \mathscr{M}(\tau). \end{cases}$$

Since A is closed, we conclude that $(v, \phi_\delta)\phi_\delta \in \mathscr{D}(A)$ and

$$A((v, \phi_\delta)\phi_\delta) = \begin{cases} (Av, \psi_\delta)\psi_\delta & \text{if } \delta \in \mathscr{M}(\tau), \\ 0 & \text{if } \delta \notin \mathscr{M}(\tau). \end{cases}$$

Since A is injective with dense domain, the left hand side is nonzero for certain $v \in \mathscr{D}(A)$. Hence $\delta \in \mathscr{M}(\tau)$, $\phi_\delta \in \mathscr{D}(A)$ and (4.2) and (4.1) hold for certain nonzero c_δ. Finally, since A is closed with dense range, $\mathscr{M}(\sigma) = \mathscr{M}(\tau)$. □

LEMMA 4.4. *Let A be a possibly unbounded, not necessarily closed, injective linear operator from $\mathscr{H}(\sigma)$ to $\mathscr{H}(\tau)$ which satisfies all other properties of Definition 4.1. Suppose that $\phi_\delta \in \mathscr{D}(A)$ for all $\delta \in \mathscr{M}(\sigma)$, $\mathscr{M}(\sigma) = \mathscr{M}(\tau)$ and,* *for each* $\delta \in \mathcal{M}(\sigma)$, *there is a complex number* c_δ *such that* $(Av, \psi_\delta) = c_\delta(v, \phi_\delta)$ *for all* $v \in \mathcal{D}(A)$. *Then the closure* $\bar{A}$ *of* A *is one-valued and injective,* $\bar{A}$ *satisfies all properties of Definition* 4.1 *and*

$$\mathcal{D}(\bar{A}) = \{v \in \mathcal{H}(\sigma) \mid \sum_{\delta \in \mathcal{M}(\sigma)} |c_\delta(v, \phi_\delta)|^2 < \infty\}. \tag{4.3}$$

Proof. Let $\{v_n\}$ be a sequence in $\mathcal{D}(A)$ such that $v_n \to v$ in $\mathcal{H}(\sigma)$ and $Av_n \to w$ in $\mathcal{H}(\tau)$. Then, for each $\delta \in \mathcal{M}(\sigma)$,

$$(w, \psi_\delta) = \lim_{n \to \infty} (Av_n, \psi_\delta) = c_\delta \lim_{n \to \infty} (v_n, \phi_\delta) = c_\delta(v, \phi_\delta).$$

Hence $v = 0$ iff $w = 0$, so $\bar{A}$ is one-valued and injective.

To prove the domain invariance and intertwining property of $\bar{A}$, let

$$v \in \mathcal{D}(\bar{A}), \text{ so } v_n \to v,\ Av_n \to \bar{A}v$$

for some sequence $\{v_n\}$ in $\mathcal{D}(A)$. If $g \in G$ then

$$\sigma(g)v_n \to \sigma(g)v \text{ and } A\sigma(g)v_n = \tau(g)Av_n \to \tau(g)\bar{A}v,$$

so $\sigma(g)v \in \mathcal{D}(\bar{A})$ and $\bar{A}\sigma(g)v = \tau(g)\bar{A}v$.

Finally, to prove (4.3), first suppose that $v \in \mathcal{H}(\sigma)$ and

$$\Sigma_{\delta \in \mathcal{M}(\sigma)} |c_\delta(v, \phi_\delta)|^2 < \infty.$$

Then

$$v = \Sigma(v, \phi_\delta)\phi_\delta,\ w := \Sigma c_\delta(v, \phi_\delta)\psi_\delta \in \mathcal{H}(\tau) \text{ and } \bar{A}\phi_\delta$$
$$= c_\delta\psi_\delta, \text{ so, } w = \bar{A}v \text{ and } v \in \mathcal{D}(\bar{A}).$$

Conversely, let $v \in \mathcal{D}(\bar{A})$. Then $\bar{A}v = \Sigma(\bar{A}v, \psi_\delta)\psi_\delta = \Sigma c_\delta(v, \phi_\delta)\psi_\delta$ (note $(\bar{A}v, \psi_\delta) = c_\delta(v, \phi_\delta)$ by (4.1)). Hence $\Sigma\, |c_\delta(v, \phi_\delta)|^2 < \infty$. □

Next we will prove a criterium for Naimark relatedness of K-multiplicity free representations σ and τ in terms of the canonical matrix elements.

THEOREM 4.5. *Let* G *be an lcsc. group with compact abelian subgroup* K. *Let* σ *and* τ *be* K*-multiplicity free representations of* G. *Let* $\{\phi_\delta\}$ *and* $\{\psi_\delta\}$ *be* K*-bases of* $\mathcal{H}(\sigma)$ *and* $\mathcal{H}(\tau)$, *respectively. For each* $\delta \in \mathcal{M}(\sigma) \cap \mathcal{M}(\tau)$ *let* $0 \neq c_\delta \in \mathbf{C}$. *Then the following two statements are equivalent:*

(a) $\sigma \overset{A}{\simeq} \tau$ *and* $A\phi_\delta = c_\delta \psi_\delta, \delta \in \mathscr{M}(\sigma)$.

(b) $\mathscr{M}(\sigma) = \mathscr{M}(\tau)$ *and, for all* $\gamma, \delta \in \mathscr{M}(\sigma)$,

$$\tau_{\gamma,\delta} = C_{\gamma,\delta}\sigma_{\gamma,\delta} \tag{4.4}$$

with $C_{\gamma,\delta} = c_\gamma / c_\delta$.

If, moreover, σ *and* τ *are irreducible then* (a) *and* (b) *are also equivalent to:*

(c) *For some* $\gamma, \delta \in \mathscr{M}(\sigma) \cap \mathscr{M}(\tau)$ (4.4) *holds for some nonzero complex* $C_{\gamma,\delta}$.

Proof.

(a) $\Rightarrow$ (b): Apply Lemma 4.3. By using (4.1) we have

$$\begin{aligned} c_\gamma(\sigma(g)\phi_\delta, \phi_\gamma) &= (A\sigma(g)\phi_\delta, \psi_\gamma) = (\tau(g)A\phi_\delta, \psi_\gamma) \\ &= c_\delta(\tau(g)\psi_\delta, \psi_\gamma)\,. \end{aligned}$$

(b) $\Rightarrow$ (a): Define A on the domain $\{v \in \mathscr{H}(\sigma) \mid \Sigma \mid c_\delta(v, \phi_\delta) \mid^2 < \infty\}$ by $Av := \Sigma c_\delta(v, \phi_\delta)\psi_\delta$. Then A is injective with dense domain and range and A satisfies (4.1). We will prove that $\mathscr{D}(A)$ is G-invariant and that A is an intertwining operator. Let $v \in \mathscr{D}(A)$, $g \in G$. Then, by (4.4) and the definition of Av:

$$\begin{aligned} c_\gamma(\sigma(g)v, \phi_\gamma) &= c_\gamma \Sigma_\delta(v, \phi_\delta)\sigma_{\gamma,\delta}(g) \\ &= \Sigma_\delta c_\delta(v, \phi_\delta)\tau_{\gamma,\delta}(g) = (\tau(g)Av, \psi_\gamma)\,. \end{aligned}$$

Hence

$$\Sigma_\gamma \mid c_\gamma(\sigma(g)v, \phi_\gamma) \mid^2 = \| \tau(g)Av \|^2 < \infty\,.$$

So $\sigma(g)v \in \mathscr{D}(A)$ and $A\sigma(g)v = \tau(g)Av$. Now apply Lemma 4.4.

(c) $\Rightarrow$ (b): (σ, τ irreducible): We will first show that $\mathscr{M}(\sigma) = \mathscr{M}(\tau)$ and, for each $\beta \in \mathscr{M}(\sigma)$, $\tau_{\gamma,\beta} = C_{\gamma,\beta}\sigma_{\gamma,\beta}$ and $\tau_{\beta,\delta} = C_{\beta,\delta}\sigma_{\beta,\delta}$ for some nonzero complex $C_{\gamma,\beta}$ and $C_{\beta,\delta}$. It follows from (4.4) evaluated for $g = g_1 k g_2$ that

$$\begin{aligned} &\sum_{\beta \in \mathscr{M}(\tau)} \beta(k)\tau_{\gamma,\beta}(g_1)\tau_{\beta,\delta}(g_2) \\ &= C_{\delta,\gamma} \sum_{\beta \in \mathscr{M}(\sigma)} \beta(k)\sigma_{\gamma,\beta}(g_1)\sigma_{\beta,\delta}(g_2)\,, \quad g_1, g_2 \in G, k \in K\,. \end{aligned}$$

Both sides are absolutely and uniformly convergent Fourier series in $k \in K$. Because of Theorem 3.2 and the irreducibility of σ and τ, for each $\beta \in \mathscr{M}(\tau)$

respectively $\beta \in \mathscr{M}(\sigma)$ the Fourier coefficient at the left respectively right hand side does not vanish identically in g_1, g_2. Hence $\mathscr{M}(\sigma) = \mathscr{M}(\tau)$ and

$$\tau_{\gamma,\beta}(g_1)\tau_{\beta,\delta}(g_2) = C_{\gamma,\delta}\sigma_{\gamma,\beta}(g_1)\sigma_{\beta,\delta}(g_2).$$

This implies

$$\tau_{\gamma,\beta} = C_{\gamma,\beta}\sigma_{\gamma,\beta} \text{ and } \tau_{\beta,\delta} = C_{\beta,\delta}\sigma_{\beta,\delta} \text{ with } C_{\gamma,\beta}C_{\beta,\delta} = C_{\gamma,\delta}.$$

By repeating this argument we prove that $\tau_{\alpha,\beta} = C_{\alpha,\beta}\sigma_{\alpha,\beta}$ for all $\alpha, \beta \in \mathscr{M}(\sigma)$ and that $C_{\alpha,\beta}C_{\beta,\delta} = C_{\alpha,\delta}$, i.e. $C_{\alpha,\beta} = C_{\alpha,\delta}/C_{\beta,\delta}$. □

COROLLARY 4.6. *Let G be an lcsc. group with compact abelian subgroup K. Then Naimark relatedness is an equivalence relation in the class of K-multiplicity free representations of G.*

4.2. THE CASE $SU(1, 1)$

Consider irreducible subquotient representations of $\pi_{\xi,\lambda}$ as classified in Theorem 3.4. By comparing K-contents it follows that the only possible nontrivial Naimark equivalences are:

$$\pi_{\xi,\lambda} \simeq \pi_{\xi,\mu} (\lambda+\xi, \mu+\xi \notin \mathbf{Z}+\tfrac{1}{2}, \lambda \neq \mu)$$

and

$$\pi^+_{\xi,\lambda} \simeq \pi^+_{\xi,-\lambda}, \quad \pi^0_{\xi,\lambda} \simeq \pi^0_{\xi,-\lambda}, \quad \pi^-_{\xi,\lambda} \simeq \pi^-_{\xi,-\lambda}$$
$$(\lambda+\xi \in \mathbf{Z}+\tfrac{1}{2}, \lambda \neq 0).$$

Suppose that σ and τ are irreducible subquotient representations of $\pi_{\xi,\lambda}$ and $\pi_{\xi,\mu}$, respectively, and that $\phi_m \in \mathscr{H}(\sigma) \cap \mathscr{H}(\tau)$ for some $m \in \mathbf{Z} + \xi$. It follows from Theorem 4.5 that $\sigma \simeq \tau$ iff $\tau_{\xi,\lambda,m,m} = \pi_{\xi,\mu,m,m}$. This last identity already holds if it is valid for the restrictions to A. In view of (2.29) and (2.30) we have: $\sigma \simeq \tau$ iff

$$\phi^{(0,2m)}_{2i\lambda}(t) = \phi^{(0,2m)}_{2i\mu}(t), \quad t \in \mathbf{R}. \tag{4.5}$$

Formula (4.5) holds if $\lambda = \pm\mu$ (cf. (2.26)). Conversely, assume (4.5) and expand both sides of (4.5) as a power series in $-(sh\ t)^2$ by using (2.23) and (2.20). The coefficients of $-(sh\ t)^2$ yield the equality

$$(m+1+\lambda)(m+1-\lambda) = (m+1+\mu)(m+1-\mu)$$

Hence $\lambda = \pm\mu$. We have proved:

THEOREM 4.7. *Let* σ *and* $\tau(\sigma \neq \tau)$ *be irreducible subquotient representations of the principal series. Then* σ *is Naimark equivalent to* τ *in precisely the following situations (cf. the notation of Theorem 3.4):*

(a) $$\pi_{\xi,\lambda} \simeq \pi_{\xi,-\lambda} (\lambda+\xi \notin \mathbf{Z}+\tfrac{1}{2}, \lambda \neq 0)$$

(b) $$\pi^{+}_{\xi,\lambda} \simeq \pi^{+}_{\xi,-\lambda}, \pi^{0}_{\xi,\lambda} \simeq \pi^{0}_{\xi,-\lambda}, \pi^{-}_{\xi,\lambda} \simeq \pi^{-}_{\xi,-\lambda} \ (\lambda+\xi \in \mathbf{Z}+\tfrac{1}{2}, \lambda \neq 0).$$

Remark 4.8. It follows from Theorem 3.4 and Theorem 4.7 that each irreducible subquotient representation of some $\pi_{\xi,\lambda}$ is Naimark equivalent to some irreducible subrepresentation of some $\pi_{\xi,\lambda}$.

It follows from Theorems 4.7 and 4.5 that for each $\xi \in \{0, \frac{1}{2}\}$ and $\lambda \in \mathbf{C}\backslash\{0\}$ we have identities

(4.6) $$\pi_{\xi,-\lambda,m,n} = C_{\xi,\lambda,m,n}\, \pi_{\xi,\lambda,m,n}$$

for certain nonzero complex constants $C_{\xi,\lambda,m,n}$, where $m, n \in \mathbf{Z} + \xi$ and, if $\lambda + \xi \in \mathbf{Z} + \frac{1}{2}$, we have the further restriction that $m, n \in (-\infty, -|\lambda| - \frac{1}{2}]$ or $m, n \in [-|\lambda| + \frac{1}{2}, |\lambda| - \frac{1}{2}]$ or $m, n \in [|\lambda| + \frac{1}{2}, \infty)$. Indeed, it follows from (2.29) and (2.26) that (4.6) holds with

(4.7) $$C_{\xi,\lambda,m,n} = \frac{c_{\xi,-\lambda,m,n}}{c_{\xi,\lambda,m,n}}.$$

A calculation using (4.7) and (2.30) shows that

(4.8) $$C_{\xi,\lambda,m,n} = c_{\xi,\lambda,m}/c_{\xi,\lambda,n}$$

with

(4.9) $$\begin{aligned} c_{\xi,\lambda,m} &= \text{const.}\, \frac{\Gamma(-\lambda+m+\frac{1}{2})}{\Gamma(\lambda+m+\frac{1}{2})} = \text{const.}\, \frac{\Gamma(-\lambda-m+\frac{1}{2})}{\Gamma(\lambda-m+\frac{1}{2})} \\ &= \text{const.}\, (-1)^{m-\xi}\, \Gamma(-\lambda+m+\tfrac{1}{2})\, \Gamma(-\lambda-m+\tfrac{1}{2}) \\ &= \text{const.}\, \frac{(-1)^{m-\xi}}{\Gamma(\lambda+m+\frac{1}{2})\, \Gamma(\lambda-m+\frac{1}{2})}. \end{aligned}$$

If $\lambda + \xi \notin \mathbf{Z} + \frac{1}{2}$ then we can use all alternatives for $c_{\xi,\lambda,m}$, but if $\lambda + \xi \in \mathbf{Z} + \frac{1}{2}$ then we can use precisely one alternative. Now, by Theorem 4.5, we obtain:

PROPOSITION 4.9. *Let* $\sigma \overset{A}{\simeq} \tau$ *be one of the equivalences of Theorem 4.7 with* σ *being a subquotient representation of* $\pi_{\xi,\lambda}$. *Then*

(4.10) $$A\phi_m = c_{\xi,\lambda,m}\,\phi_m,$$

where $m \in \mathbf{Z} + \xi$ *such that* $\delta_m \in \mathscr{M}(\sigma)$ *and* $c_{\xi,\lambda,m}$ *is given by (4.9).*

4.3. Notes

4.3.1. Definition 4.1 of Naimark relatedness goes back to NAIMARK [33]. He introduced this concept in the context of representations of the Lorentz group on a reflexive Banach space. Next he gave a much more involved definition in his book [34, Ch. 3, §9, No. 3]. Afterwards, many different versions of this definition appeared in literature, which all refer to [34]. We mention ZELOBENKO & NAIMARK [51, Def. 2] ("weak equivalence" for representations on locally convex spaces), FELL [13, §6] (Naimark relatedness for "linear system representations") and WARNER [48, p. 232 and p. 242]. Warner starts with the definition of Naimark relatedness for Banach representations of an associative algebra over $\mathbf{C}$ (this definition is similar to our Definition 4.1) and next he defines Naimark relatedness for Banach representations of an lcsc. group G in terms of Naimark relatedness for the corresponding representations of $M_c(G)$ or (equivalently) $C_c(G)$. Warner's definition seems to be standard now. POULSEN [35, Def. 33] gives Naimark's original definition [33] and he calls it weak equivalence. FELL [13] (see also WARNER [48, Theorem 4.5.5.2]) proved that, for K-finite Banach representations of a connected unimodular Lie group, two representations are Naimark related iff they are infinitesimally equivalent.

4.3.2. Our implication (*c*) $\Rightarrow$ (*a*) in Theorem 4.5 is related to WALLACH [44, Cor. 2.1]. Theorem 4.7 can be formulated for general semisimple Lie groups G. If $\pi_{\xi,\lambda}$ is an irreducible principal series representation and if $s \in W$ then $\pi_{\xi,\lambda} \simeq \pi_{\xi^s, s\cdot\lambda}$ (cf. WALLACH [44, Theorem 3.1]). This yields part (a). Regarding part (b) see LEPOWSKY's [29, Theorem 9.8] result that $\pi_{\xi,\lambda}$ and $\pi_{\xi^s, s\cdot\lambda}$ have equivalent composition series.

4.3.3. Theorem 4.7 was first proved in the unitarizable cases by BARGMANN [2]. He used infinitesimal methods. TAKAHASHI [39] proved Theorem 4.7 (again in the unitarizable cases) by calculating the diagonal matrix elements $\pi_{\xi,\lambda,m,n}(a_t)$ and by observing that they are even in λ. GELFAND, GRAEV & VILENKIN [17, Ch. VII, §4] obtained Theorem 4.7 by working in the noncompact realization of the principal series and by explicitly constructing all possible intertwining operators.

4.3.4. Analogues of the results in §4.1 hold for nonabelian K and (in Lemmas 4.3, 4.4 and Corollary 4.6) for K-finite representations, cf. [27, §4].

5. Equivalence of irreducible representations of $SU(1,1)$ to subrepresentations of the principal series

The first two subsections review some generalities about Gelfand pairs and spherical functions. By using the concepts developed there we can next, in §5.4, translate the problem of classifying the irreducible representations of $SU(1, 1)$ in such a way that the problem can be solved by global methods. For this the generalized Abel transform (§5.3) and the Chebyshev transform pair of Deans (Theorem 5.10) are the main tools. The problem is finally reduced to finding the continuous characters on the convolution algebra $\mathscr{D}_{\text{even}}(\mathbf{R})$ (Prop. 5.7).

5.1. Spherical functions

We remember some of the standard facts about spherical functions (cf. for instance Godement [20], Helgason [25, Ch. X], Faraut [12, Ch. 1]). Let G be a unimodular lcsc. group with compact subgroup K. (G, K) is called a *Gelfand pair* if $C_c(K\backslash G/K)$ is a commutative algebra under convolution. If there is a continuous involutive automorphism α on G such that $\alpha(KxK) = Kx^{-1}K (x \in G)$ then (G, K) is a Gelfand pair. If (G, K) is a Gelfand pair and the irreducible representation τ of G is unitary or K-finite then the representation 1 of K has multiplicity 0 or 1 in τ.

Let (G, K) be a Gelfand pair. A *spherical function* is a function $\phi \neq 0$ on G such that

$$\phi(x)\phi(y) = \int_K \phi(xky)dk, \quad x, y \in G\,.$$

The nonzero continuous algebra homomorphisms from $C_c(K\backslash G/K)$ (or $C_c^\infty(K\backslash G/K)$ if G is a Lie group) to $\mathbf{C}$ are precisely of the form

$$f \to \int_G f(x)\phi(x^{-1})dx\,, \tag{5.1}$$

where ϕ is a spherical function. If τ is a K-unitary representation of G and if $\mathscr{H}(\tau)$ contains a K-fixed unit vector v, unique up to a constant factor, then $x \to (\tau(x)v, v)$ is a spherical function.

5.2. Spherical functions of type δ

Let G be a unimodular lcsc. group with compact subgroup K. Let

$$K^* := \{(k, k) \in G \times K \mid k \in K\} .$$

Let $\delta \in \hat{K}$ and let τ be a K-unitary representation of G. Then $\tau \otimes \check{\delta}$ ($\check{\delta}$ the contragredient representation to δ) is a K^*-unitary representation of $G \times K$ on $\mathscr{H}(\tau) \otimes \mathscr{H}(\delta)$.

LEMMA 5.1. *The multiplicity of* δ *in* $\tau|_K$ *is equal to the multiplicity of the representation* 1 *of* K^* *in* $\tau \otimes \check{\delta}|_{K^*}$. τ *is irreducible iff* $\tau \otimes \check{\delta}$ *is irreducible.* τ *is unitary iff* $\tau \otimes \check{\delta}$ *is unitary.*

This can be proved immediately. By using the results summarized in §5.1 we conclude that $(G \times K, K^*)$ is a Gelfand pair if there exists a continuous involutive homomorphism α on G such that for each $(g, k) \in G \times K$ we have $\alpha(g) = k_1 g^{-1} k_2$, $\alpha(k) = k_1 k^{-1} k_2$ for certain $k_1, k_2 \in K$. Furthermore, if $(G \times K, K^*)$ is a Gelfand pair and if the irreducible representation τ of G is unitary or K-finite then τ is K-multiplicity free. In particular, this applies to $SU(1, 1)$:

PROPOSITION 5.2. *If* $G = SU(1, 1)$ *then* $(G \times K, K^*)$ *is a Gelfand pair.*

Proof. For $g \in SU(1, 1)$ define $\alpha(g) := {}^t(g^{-1})$. Then α is a continuous involutive automorphism on G and $\alpha(a_t) = a_{-t}$ on A, $\alpha(u_\theta) = u_{-\theta}$ on K. Since $G = KAK$, α has the required properties. □

Let $(G \times K, K^*)$ be a Gelfand pair. Identify $G \times \{e\}$ with G. A spherical function on $G \times K$ is completely determined by its restriction to G. By using the results mentioned in §5.1 we obtain the following properties. First, a continuous function ϕ on G is the restriction to G of a spherical function on $G \times K$ iff $\phi \neq 0$ and

$$\phi(x)\phi(y) = \int_K \phi(xkyk^{-1})dk, \qquad x, y \in G .$$

Next, let

$$I_c(G) \text{ (or } I_c^\infty(G)) := \{f \in C_c(G) \text{ (or } C_c^\infty(G)) \mid f(kgk^{-1}) = f(g),\ g \in G, k \in K\} .$$

These are commutative topological algebras under convolution and their characters are precisely of the form (5.1), where ϕ is a spherical function on $G \times K$. If ϕ is a spherical function on $G \times K$ then there is a $\delta \in \hat{K}$ such that for all $x \in G$ the function $k \to \phi(xk)$ on K belongs to δ. Then δ is called a *spherical function of type* δ on G (with respect to K), cf. GODEMENT [19]. It is funny that spherical functions of type δ are on the one hand generalizations of ordinary spherical functions for (G, K), on the other hand restrictions to G of ordinary spherical functions for $(G \times K, K^*)$.

For convenience, we take a one-dimensional $\delta \in \hat{K}$. Then a spherical function ϕ on $G \times K$ is of type δ iff

$$\phi(xk) = \phi(kx) = \delta(k)\phi(x), \qquad x \in G, k \in K .$$

Let

$$\begin{aligned} &I_{c,\delta}(G) \ \left(\text{or } I^{\infty}_{c,\delta}(G)\right) \\ &:= \left\{f \in C_c(G) \ \left(\text{or } C^{\infty}_c(G)\right) \mid f(xk) = f(kx) \right. \\ &\qquad \left. = \delta(k)f(x),\ x \in G,\ k \in K\right\} . \end{aligned}$$

These are closed subalgebras of $I_c(G)$ (or $I^{\infty}_c(G)$) and their characters are precisely of the form (5.1), where ϕ is a spherical function of type δ. Finally, if τ is a K-unitary representation of G and if $\mathscr{H}(\tau)$ contains a unit vector v satisfying $\tau(k)v = \delta(k)v$, unique up to a constant factor, then $x \to (\tau(x)v, v)$ is a spherical function of type δ.

5.3. THE GENERALIZED ABEL TRANSFORM

Let G be a connected noncompact real semisimple Lie group with finite center. Use the notation of §2.2. For given Haar measures dk, da, dn on K, A, N, respectively, normalize the Haar measure on G such that

$$(5.2)\qquad \int_G f(g)dg = \int_{K \times A \times N} f(kan)e^{2\rho(\log a)}\, dk\, da\, dn,\ f \in C_c(G)$$

(cf. HELGASON [25, Ch. X, Prop. 1.11]). Note the property

$$(5.3)\qquad \int_N f(n)dn = e^{2\rho(\log a)} \int_N f(ana^{-1})dn,\ f \in C_c(N),\ a \in A$$

(cf. [25, Ch. X, proof of Prop. 1.11]).

For $\lambda \in \mathfrak{a}_{\mathbf{C}}^*$ let U^λ be the representation of G induced by the one-dimensional representation $an \to e^{\lambda(\log a)}$ of the subgroup AN:

$$(5.4)\quad \big(U^\lambda(g)f\big)(k) := e^{-(\rho+\lambda)H(g^{-1}k)}\, f\big(u(g^{-1}k)\big),\ f \in L^2(K),\ g \in G,\ k \in K\,.$$

The representation U^λ is easily seen to split as a direct sum of principal series representations $\pi_{\xi,\lambda}$. U^λ restricted to K is the left regular representation of K.

Let $\delta \in \hat{K}$. For convenience, suppose that δ is one-dimensional. The *generalized Abel transform* $f \to F_f^\delta : I_{c,\delta}(G) \to C_c(A)$ is defined by

$$(5.5)\qquad F_f^\delta(a) := e^{\rho(\log a)} \int_N f(an)dn,\ a \in A\,.$$

If $G = SU(1, 1)$ and $\delta = 1$ then this transform can be rewritten as the classical Abel transform, cf. §5.4.

PROPOSITION 5.3. *The mapping $f \to F_f^\delta$ is a continuous homomorphism (with respect to convolution on G and A, respectively) from $I_{c,\delta}^\infty(G)$ to $C_c^\infty(A)$. Furthermore,*

$$(5.6)\quad \int_A F_f^\delta(a)e^{-\lambda(\log a)}\,da = \int_G f(g)\big(U^\lambda(g^{-1})\check{\delta}, \check{\delta}\big)dg,\ f \in I_{c,\delta}^\infty(G),\ \lambda \in \mathfrak{a}_{\mathbf{C}}^*\,,$$

where $(.,.)$ denotes the inner product on $L^2(K)$.

Proof. The continuity is immediate. The homomorphism property follows easily from (5.2) and (5.3) (cf. WARNER [49, pp. 34, 35]). For the proof of (5.6) substitute (5.4) into the right hand side of (5.6):

$$\begin{aligned}
\int_G f(g)\big(U^\lambda(g^{-1})\check{\delta}, \check{\delta}\big)dg &= \int_G\int_K f(g)e^{-(\rho+\lambda)H(gk)}\,\delta\big((u(gk))^{-1}k\big)dk\,dg\\
&= \int_G f(g)e^{-(\rho+\lambda)H(g)}\,\delta\big((u(g))^{-1}\big)dg\\
&= \int_{K\times A\times N} f(kan)e^{(\rho-\lambda)\log a}\,\delta(k^{-1})dk\,da\,dn\\
&= \int_A\int_N f(an)e^{(\rho-\lambda)\log a}\,dn\,da\\
&= \int_A F_f^\delta(a)e^{-\lambda(\log a)}\,da\,.
\end{aligned}$$

$\square$

Now let $G = SU(1, 1)$. Write $F_f^n(t)$ and $I_{c,n}^{\infty}(G)$ instead of $F_f^{\delta_n}(a_t)$ and $I_{c,\delta_n}^{\infty}(G)$, respectively. If $n \in \mathbf{Z} + \xi$ then (5.5) and (5.6) take the form

$$F_f^n(t) = e^{\frac{1}{2}t} \int_{-\infty}^{\infty} f(a_t n_z) dz \tag{5.7}$$

and

$$\int_{-\infty}^{\infty} F_f^n(t) e^{-\lambda t}\, dt = \int_G f(g) \pi_{\xi,\lambda,n,n}(g^{-1}) dg,\ f \in I_{c,n}^{\infty}(G),\ \lambda \in \mathbf{C}\,, \tag{5.8}$$

where $dg = (2\pi)^{-1} e^t d\theta dt dz$ if $g = u_\theta a_t n_z$.

5.4. The main theorem

It is the purpose of this section to prove:

Theorem 5.4. *Let τ be an irreducible K-unitary representation of $SU(1, 1)$ which is K-finite or unitary. Then τ is Naimark equivalent to an irreducible subrepresentation of some principal series representation $\pi_{\xi,\lambda}$.*

By Proposition 5.2 τ is K-multiplicity free. If $\delta_n \in \mathcal{M}(\tau)$ then write $\tau_{n,n}$ instead of τ_{δ_n,δ_n}. In view of Theorem 4.5 and Remark 4.8 it is sufficient for the proof of Theorem 5.4 to show that for some $\delta_n \in \mathcal{M}(\tau)$, for some $\lambda \in \mathbf{C}$ and for $\xi \in \{0, \frac{1}{2}\}$ with $n \in \mathbf{Z} + \xi$ we have

$$\tau_{n,n} = \pi_{\xi,\lambda,n,n}\,. \tag{5.9}$$

Both sides of (5.9) are spherical functions of type δ_n. Then (5.9) holds if the corresponding characters on $I_{c,n}^{\infty}(G)$ are equal. Hence Theorem 5.4 will follow from

Proposition 5.5. *Let $G = SU(1, 1)$, $n \in \frac{1}{2}\mathbf{Z}$. Let α be a continuous character on $I_{c,n}^{\infty}(G)$. Then*

$$\alpha(f) = \int_G f(g) \pi_{\xi,\lambda,n,n}(g^{-1}) dg,\ f \in I_{c,n}^{\infty}(G)\,, \tag{5.10}$$

for some $\lambda \in \mathbf{C}$ and for $\xi \in \{0, \frac{1}{2}\}$ such that $n \in \mathbf{Z} + \xi$.

Now substitute (5.8) into the right hand side of (5.10). Thus, for the proof of Prop. 5.5 we have to show that each continuous character α on $I_{c,n}^{\infty}(G)$ takes the form

$$\alpha(f) = \int_{-\infty}^{\infty} F_f^n(t) e^{-\lambda t}\, dt,\ f \in I_{c,n}^{\infty}(G)\,. \tag{5.11}$$

for some $\lambda \in \mathbf{C}$. In §5.5 we will prove:

THEOREM 5.6. *Let* $G = SU(1, 1)$, $n \in \frac{1}{2}\mathbf{Z}$. *The mapping* $f \to F_f^n$ *is a topological algebra isomorphism from* $I_{c,n}^{\infty}(G)$ *onto* $\mathscr{D}_{\text{even}}(\mathbf{R})$, *the algebra of even* C^{∞}*-functions with compact support on* $\mathbf{R}$.

Thus, in view of (5.11) we are left to prove:

PROPOSITION 5.7. *The continuous characters on* $\mathscr{D}_{\text{even}}(\mathbf{R})$ *have the form*

$$h \to \int_{-\infty}^{\infty} h(t) e^{-\lambda t}\, dt$$

for some $\lambda \in \mathbf{C}$.

5.5. COMPLETION OF THE PROOF OF THE MAIN THEOREM

By the discussion in §5.4 we reduced the proof of Theorem 5.4 to the task of proving Theorem 5.6 and Prop. 5.7. Theorem 5.6 was partly proved in Prop. 5.3. It is left to prove that $f \to F_f^n$ is injective on $I_{c,n}^{\infty}(G)$ with image $\mathscr{D}_{\text{even}}(\mathbf{R})$ and that the inverse mapping is continuous. In order to establish this we identify both $I_{c,n}^{\infty}(G)$ and $\mathscr{D}_{\text{even}}(\mathbf{R})$, considered as topological vector spaces, with $\mathscr{D}([1, \infty))$ and we rewrite $f \to F_f^n$ as a mapping from $\mathscr{D}([1, \infty))$ onto itself. This mapping turns out to be a known integral transformation, for which an inverse transformation can be explicitly given. First note:

LEMMA 5.8. *The formula*

$$f(x) = g(x^2) \tag{5.12}$$

defines an isomorphism of topological vector spaces $f \to g$ *from* $\mathscr{D}_{\text{even}}(\mathbf{R})$ *onto* $\mathscr{D}([0, \infty))$.

Proof. Clearly, if $g \in \mathscr{D}([0, \infty))$ then $f \in \mathscr{D}_{\text{even}}(\mathbf{R})$ and the mapping $g \to f$ is continuous. Conversely, let $f \in \mathscr{D}_{\text{even}}(\mathbf{R})$ and let g be defined by (5.12). By complete induction with respect to n we prove: $g^{(n)}(0)$ exists and there is a function $f_n \in \mathscr{D}_{\text{even}}(\mathbf{R})$ such that

$$f_n(x) = g^{(n)}(x^2),\ x \in \mathbf{R},$$

and $f \to f_n : \mathscr{D}_{\text{even}}(\mathbf{R}) \to \mathscr{D}_{\text{even}}(\mathbf{R})$ is continuous. Indeed, suppose this is proved up to $n - 1$. Then

$$2x(g^{(n-1)})'(x^2) = f'_{n-1}(x) = \int_0^x f''_{n-1}(y)dy,$$

so

$$g^{(n)}(x^2) = \tfrac{1}{2}\int_0^1 f''_{n-1}(tx)dt =: f_n(x). \qquad \square$$

For $f \in I^{\infty}_{c,n}(G)$ define

$$\tilde{f}(x) := f\left(\begin{pmatrix} x & (x^2-1)^{\frac{1}{2}} \\ (x^2-1)^{\frac{1}{2}} & x \end{pmatrix}\right),\ x \in [1, \infty). \tag{5.13}$$

For $f \in I^{\infty}_{c,n}(G)$ define

$$\tilde{h}(ch\tfrac{1}{2}t) := h(t),\ t \in \mathbf{R}. \tag{5.14}$$

LEMMA 5.9. *The mapping* $f \to \tilde{f}$ *defined by* (5.13) *is an isomorphism of topological vector spaces from* $I^{\infty}_{c,n}(G)$ *onto* $\mathscr{D}([1, \infty))$. *The mapping* $h \to \tilde{h}$ *defined by* (5.35) *is an isomorphism of topological vector spaces from* $\mathscr{D}_{\text{even}}(\mathbf{R})$ *onto* $\mathscr{D}([1, \infty))$.

Proof. The second statement follows from Lemma 5.8. For the proof of the first statement introduce global real analytic coordinates on G by the mapping

$$(z, \phi) \to \begin{pmatrix} e^{\frac{1}{2}i\phi}(1+|z|^2)^{\frac{1}{2}} & z \\ \bar{z} & e^{-\frac{1}{2}i\phi}(1+|z|^2)^{\frac{1}{2}} \end{pmatrix}$$

from $C \times (\mathbf{R}/4\pi\mathbf{Z})$ onto G. If $g \in \mathscr{D}([1, \infty))$ and

$$f\left(\begin{pmatrix} e^{\frac{1}{2}i\phi}(1+|z|^2)^{\frac{1}{2}} & z \\ \bar{z} & e^{-\frac{1}{2}i\phi}(1+|z|^2)^{\frac{1}{2}} \end{pmatrix}\right) := e^{in\phi}g((1+|z|^2)^{\frac{1}{2}})$$

then $f \in I^{\infty}_{c,n}(G)$, $\tilde{f} = g$ and the mapping $g \to f$ is continuous. Conversely, if $f \in I^{\infty}_{c,n}(G)$ then f, as a function of z and ϕ, is radial in z, so the function

$$z \to f\begin{pmatrix} (1+z^2)^{\frac{1}{2}} & z \\ z & (1+z^2)^{\frac{1}{2}} \end{pmatrix}, z \in \mathbf{R},$$

belongs to $\mathscr{D}_{\text{even}}(\mathbf{R})$. Now make the transformation $z = (x^2-1)^{\frac{1}{2}}$ and apply Lemma 5.8. It follows that $\tilde{f} \in \mathscr{D}([1, \infty))$ and that the mapping $f \to \tilde{f}$ is continuous. □

Define the Chebyshev polynomial $T_n(x)$ by

$$T_n(\cos\theta) := \cos n\theta . \tag{5.15}$$

It follows from (5.7) that, for $f \in I^{\infty}_{c,n}(G)$:

$$\begin{aligned} F^n_f(t) &= e^{\frac{1}{2}t} \int_{-\infty}^{\infty} f\left(\begin{pmatrix} ch\frac{1}{2}t + \frac{1}{2}ize^{\frac{1}{2}t} & * \\ * & * \end{pmatrix}\right) dz \\ &= e^{\frac{1}{2}t} \int_{-\infty}^{\infty} \tilde{f}(|ch\tfrac{1}{2}t + \tfrac{1}{2}ize^{\frac{1}{2}t}|)\left(\frac{ch\frac{1}{2}t + \frac{1}{2}ize^{\frac{1}{2}t}}{|ch\frac{1}{2}t + \frac{1}{2}ize^{\frac{1}{2}t}|}\right)^{2n} dz \\ &= e^{\frac{1}{2}t} \int_{0}^{\infty} \tilde{f}(|ch\tfrac{1}{2}t + \tfrac{1}{2}ize^{\frac{1}{2}t}|) T_{2|n|}\left(\frac{ch\frac{1}{2}t}{|ch\frac{1}{2}t + \frac{1}{2}ize^{\frac{1}{2}t}|}\right) dz , \end{aligned}$$

so

$$F^n_f(t) = 2 \int_{ch\frac{1}{2}t}^{\infty} \tilde{f}(y) T_{2|n|}(y^{-1} ch\tfrac{1}{2}t)(y^2 - ch^2\tfrac{1}{2}t)^{-\frac{1}{2}} y\,dy .$$

This formula shows that F^n_f is even on $\mathbf{R}$, so $F^n_f \in \mathscr{D}_{\text{even}}(\mathbf{R})$. Now, by (5.14):

$$\tilde{F}^n_f(x) = 2 \int_{x}^{\infty} \tilde{f}(y) T_{2|n|}(y^{-1}x)(y^2 - x^2)^{-\frac{1}{2}} y\,dy, \; x \in [1, \infty] . \tag{5.16}$$

For $n = 0$, (5.16) takes the form

$$\tilde{F}^0_f(x) = 2 \int_{x}^{\infty} \tilde{f}(y)(y^2 - x^2)^{-\frac{1}{2}} y\,dy ;$$

The problem of inverting this just means to solve the Abel integral equation, as was pointed out by GODEMENT [20]. Indeed, we get

$$\tilde{f}(y) = -\pi^{-1} \int_y^\infty \frac{d}{dx} \tilde{F}_f^0(x)(x^2-y^2)^{-\frac{1}{2}} dx .$$

For general n, we can use an inversion formula obtained by DEANS [7, (30)], see also MATSUSHITA [30, §2.3] and KOORNWINDER [27, §5.9]:

THEOREM 5.10. *For* $m = 0, 1, 2, \ldots, g \in \mathscr{D}([1, \infty)), x \in [1, \infty)$ *define*

$$(A_m g)(x) := 2 \int_x^\infty g(y) T_m(y^2-x^2)^{-\frac{1}{2}} y dy , \tag{5.17}$$

$$(B_m g)(x) := -\pi^{-1} \int_x^\infty g'(y) T_m(x^{-1}y)(y^2-x^2)^{-\frac{1}{2}} dy . \tag{5.18}$$

Then A_m *and* B_m *map* $\mathscr{D}([1, \infty))$ *into itself and* $A_m B_m = id, B_m A_m = id.$

This theorem shows that $f \to F_f^n$ is a linear bijection from $I_{c,n}^\infty(G)$ onto $\mathscr{D}_{\text{even}}(\mathbf{R})$. Finally in order to prove the continuity of the inverse mapping, we show that B_m is continuous. Just expand $T_m(x^{-1}y)$ as a polynomial and use that

$$\left(x^{-1} \frac{d}{dy}\right)^p \int_x^\infty h(y)(y^2-x^2)^{-\frac{1}{2}} y dy$$

$$= \int_x^\infty \left(y^{-1} \frac{d}{dy}\right)^p h(y)(y^2-x^2)^{-\frac{1}{2}} y dy$$

by the properties of the Weyl fractional integral transform (cf. [11, Ch. 13]). This completes the proof of Theorem 5.6.

Proof of proposition 5.7. Extend α to a continuous linear functional on $\mathscr{D}(\mathbf{R})$, for instance by putting $\alpha(f) = 0$ if f is odd. Choose $f_1 \in \mathscr{D}_{\text{even}}(\mathbf{R})$ such that $\alpha(f_1) \neq 0$. Let

$$(\lambda(y) f_1)(x) := f_1(x-y), \; x, y \in \mathbf{R} .$$

By the continuity and homomorphism property of α we have, for $f \in \mathscr{D}_{\mathrm{even}}(\mathbf{R})$:

$$\alpha(f_1)\alpha(f) = \alpha(f_1 * f) = \int_{-\infty}^{\infty} \alpha(\lambda(y)f_1)f(y)dy .$$

Hence

$$\alpha(f) = \int_{-\infty}^{\infty} f(y)\beta(y)dy, \; f \in \mathscr{D}_{\mathrm{even}}(\mathbf{R}) ,$$

where

$$\beta(y) := \tfrac{1}{2}(\alpha(f_1))^{-1}(\alpha(\lambda(y)f_1) + \alpha(\lambda(-y)f_1)) .$$

Then β is even and it is a continuous function by the continuity of α. It follows from the homomorphism property of α and from the fact that β is even, that

$$\beta(x)\beta(y) = \tfrac{1}{2}(\beta(x+y) + \beta(x-y)) ,$$

so $\beta(0) = 1$. This is d'Alembert's functional equation. By continuity, $Re\, \beta(x) > 0$ if $0 \leqslant x \leqslant x_0$ for some $x_0 > 0$. Then $\beta(x_0) = \cosh c$ for some complex $c = a + ib$ with $a \geqslant 0$, $-\frac{1}{2}\pi < b < \frac{1}{2}\pi$. Now, following the proof in ACZEL [1, 2.4.1] it can be shown [1]) that for all integer $n, m \geqslant 0$

$$\beta\left(\frac{n}{2^m} x_0\right) = \cosh\left(\frac{c}{x_0} \frac{n}{2^m} x_0\right) .$$

So, by continuity and evenness of β:

$$\beta(x) = \cosh\left(\frac{c}{x_0} x\right) \text{ for all } x \in \mathbf{R} . \qquad \square$$

5.6. NOTES

5.6.1. Some other examples of Gelfand pairs $(G \times K, K^*)$ are provided by $G = SO_0(n, 1)$, $K = SO(n)$ and $G = SU(n, 1)$, $K = S(U(n) \times U(1))$, cf. BOERNER [4, Ch. VII, §12; Ch. V, §6], DIXMIER [8] or KOORNWINDER [27, Theorems 5.7, 5.8].

5.6.2. The main Theorem 5.4, which was first proved in the case of unitary representations by BARGMANN [2], is a special case of the *subrepresentation*

[1]) I thank H. van Haeringen for this reference.

theorem for noncompact semisimple Lie groups due to Casselman (cf. WALLACH [47, Cor. 7.5]). Casselman's theorem improves HARISH-CHANDRA'S [22, Theorem 4] *subquotient theorem.*

5.6.3. The generalized Abel transform $f \to F_f^\delta$ can be defined for general K-type δ. It was introduced by HARISH-CHANDRA [24, p. 595] in the spherical case, TAKAHASHI [40, §2] in the case $G = SO_0(n, 1)$ and WARNER [49, 6.2.2] in the general case. The injectivity of this transform holds generally, cf. WARNER [49]. The image of $I_{c,\delta}^\infty(G)$ under this transform is known in the spherical case (cf. GANGOLLI [16]) and if G has real rank 1 and δ is one-dimensional (cf. WALLACH [46]), but seems to be unknown in the general case (cf. WARNER [49, p. 36]).

5.6.4. In [39] TAKAHASHI also reduces the proof of Theorem 5.4 to Proposition 5.5. However, he proves Prop. 5.5 by considering eigenfunctions of the Casimir operator, since he did not know, then, how to invert the transform $f \to F_f^n$. In [42] he independently obtained a proof of Prop. 5.5 similar to ours. Earlier, in [40, §4.1] he used a similar method in the spherical case of $G = SO_0(n, 1)$. NAIMARK [34, Ch. 3, §9] proved the subquotient theorem for $SL(2, \mathbf{C})$ by methods somewhat related to ours.

5.6.5. Part of Lemma 5.8 is contained in WHITNEY [50]. See SCHWARZ [37] for a theorem on C^∞-functions which are invariant under a more general Weyl group.

5.6.6. Theorem 5.10 more generally holds with Gegenbauer polynomials of integer of half integer order as kernels, cf. DEANS [6], [7], KOORNWINDER [27, §5.9]. Deans' proof uses the inversion formula for the Radon transform. The author's proof uses Weyl fractional integral transforms and generalized fractional integral transforms studied by SPRINKHUIZEN [38]. MATSUSHITA [30, §2.3] considers the transformation $f \to F_f^n$ for general real n in the context of the universal covering group of $SL(2, \mathbf{R})$ and he derives the inversion formula with a proof due to T. Shintani, which uses Mellin transforms.

6. UNITARIZABILITY OF IRREDUCIBLE SUBREPRESENTATIONS OF THE PRINCIPAL SERIES

6.1. A CRITERIUM FOR UNITARIZABILITY

Remember that a representation of an lcsc. group G on a Hilbert space is strongly continuous if and only if it is weakly continuous (cf. WARNER [48, Prop. 4.2.2.1]). Thus, if τ is a (strongly continuous) Hilbert representation of G then $\tilde{\tau}$ defined by

(6.1) $$\tilde{\tau}(g) := \tau(g^{-1})^*,\ g \in G\,,$$

is again a (strongly continuous) Hilbert representation of G on $\mathscr{H}(\tau)$. The representation $\tilde{\tau}$ is called the *conjugate contragredient* to τ. The representation τ is unitary iff $\tilde{\tau} = \tau$.

THEOREM 6.1. *Let G be an lcsc. group with compact abelian subgroup K. Let τ be a K-multiplicity free representation of G. Let $\{\phi_\delta\}$ be a K-basis for $\mathscr{H}(\tau)$. Let $c_\delta(\delta \in \mathscr{M}(\tau))$ be positive real numbers. Then the following statements are equivalent to each other:*

(a) *τ is Naimark equivalent to some unitary representation.*

(b) $\tau \overset{A}{\simeq} \tilde{\tau}$ *with* $A\phi_\delta = c_\delta\phi_\delta(\delta \in \mathscr{M}(\tau))$.

(c) $\overline{\tau_{\gamma,\delta}(g^{-1})} = \dfrac{c_\delta}{c_\gamma}\,\tau_{\delta,\gamma}(g),\ \gamma,\ \delta \in \mathscr{M}(\tau),\ g \in G.$

If, moreover, τ is irreducible then (a), (b) *and* (c) *are equivalent to:*

(d) *For some $\delta \in \mathscr{M}(\tau)$ we have*

$$\overline{\tau_{\gamma,\delta}(g^{-1})} = \frac{c_\delta}{c_\gamma}\,\tau_{\delta,\gamma}(g),\ g \in G, \textit{for all}\ \ \gamma \in \mathscr{M}(\tau)\,.$$

If (b) *holds then $\tau(g)(g \in G)$ is unitary with respect to a new inner product $<\cdot,\cdot>$ on $\mathscr{D}(A)$ defined by*

(6.2) $$<\phi_\gamma, \phi_\delta> := \begin{cases} 0 & \text{if } \gamma \neq \delta, \\ c_\delta & \text{if } \gamma = \delta. \end{cases}$$

Proof. First observe that $\tilde{\tau}(k)\phi_\delta = \delta(k)\phi_\delta(k \in K)$, so $\{\phi_\delta\}$ is a K-basis with respect to $\tilde{\tau}$ as well. We have

(6.3) $$\tilde{\tau}_{\gamma,\delta}(g) = \overline{\tau_{\delta,\gamma}(g^{-1})}\,.$$

<u>(a) $\Rightarrow$ (b)</u> : Let $\tau \overset{B}{\simeq} \sigma$ with σ unitary. Then $\sigma = \tilde{\sigma}$ and $\tilde{\sigma} \overset{B^*}{\simeq} \tilde{\tau}$. Let $\{\psi_\delta\}$ be a K-basis for $\mathscr{H}(\sigma)$. Let $B\phi_\delta = b_\delta\psi_\delta(\delta \in \mathscr{M}(\tau))$. Then, by Theorem 4.5:

$$\tilde{\tau}_{\gamma,\delta} = \frac{\overline{b_\gamma}}{\overline{b_\delta}}\,\sigma_{\gamma,\delta} = \left|\frac{b_\gamma}{b_\delta}\right|^2 \tau_{\gamma,\delta}\,,$$

so (b) holds.

(b) $\Rightarrow$ (a) : Assume (b). Then A is self-adjoint and positive definite. Define a new inner product $<\cdot,\cdot>$ on $\mathscr{D}(A)$ by $<v, w> := (Av, w)$. Then, for $v, w \in \mathscr{D}(A)$, $g \in G$, we have:

$$\begin{aligned} <\tau(g)v, \tau(g)w> &= \big(A\tau(g)v, \tau(g)w\big) = \big(\tilde{\tau}(g^{-1})A\tau(g)v, w\big) \\ &= \big(A\tau(g^{-1})\tau(g)v, w\big) = (Av, w) = <v, w>, \end{aligned}$$

i.e. $<\tau(g)v, \tau(g)w> = <v, w>$. Thus τ is a unitary representation on $\mathscr{D}(A)$ with respect to the new inner product. (Weak continuity of τ is easily proved.) Let σ be the extension of this representation to a unitary representation in the Hilbert space completion $\mathscr{H}(\sigma)$ of $\mathscr{D}(A)$ with respect to $<\cdot,\cdot>$. Then $\tau \overset{B}{\simeq} \sigma$, where B is the closure of the identity operator on $\mathscr{D}(A)$ (cf. Lemma 4.4). Note that we have also proved the last part of the theorem.

The equivalence of (c) or (d) with (b) follows from Theorem 4.5. □

6.2. THE CASE $SU(1, 1)$

It follows from (2.30) that

$$\overline{c_{\xi,\lambda,n,m}} = (-1)^{m-n}\, c_{\xi,-\bar{\lambda},m,n}\,. \tag{6.4}$$

Combination of (6.3), (2.29) and (6.4) yields

$$\tilde{\pi}_{\xi,\lambda} = \pi_{\xi,-\bar{\lambda}}\,. \tag{6.5}$$

In §6.1 we showed that a necessary condition for unitarizability of an irreducible subquotient representation τ of $\pi_{\xi,\lambda}$ is the equivalence of τ and $\tilde{\tau}$. In view of (6.5) and Theorem 4.7 this is only possible if $\bar{\lambda} = \pm\lambda$, that is, if λ is real or imaginary. If λ is imaginary then $\tilde{\pi}_{\xi,\lambda} = \pi_{\xi,\lambda}$, so $\pi_{\xi,\lambda}$ is already unitary. Let us now examine the case that λ is real and nonzero. Then $\tilde{\pi}_{\xi,\lambda} = \pi_{\xi,-\lambda}$. If τ is an irreducible subquotient representation of $\pi_{\xi,\lambda}$ then $\tau \overset{A}{\simeq} \tilde{\tau}$ with (cf. (4.10))

$$A\phi_m = c_{\xi,\lambda,m}\,\phi_m,\ \phi_m \in \mathscr{H}(\tau)\,, \tag{6.6}$$

where $c_{\xi,\lambda,m}$ is given by (4.9). Now a sufficient condition for the unitarizability of τ is that the coefficients $c_{\xi,\lambda,m}$ are all positive or all negative for $\phi_m \in \mathscr{H}(\tau)$. Referring to the classification in Theorem 3.4 we will examine these coefficients. (Because of equivalence, it is not necessary to treat the cases where $\lambda < 0$.)

(a) $$\pi_{0,\lambda}(\lambda>0,\ \lambda\notin\mathbf{Z}+\tfrac{1}{2}).$$

$$c_{0,\lambda,m}=\frac{(-\lambda+\frac{1}{2})_{|m|}}{(\lambda+\frac{1}{2})_{|m|}},\quad m\in\mathbf{Z}.$$

$c_{0,\lambda,m}$ has fixed sign iff $0<\lambda<\frac{1}{2}$.

(b) $$\pi_{\frac{1}{2},\lambda}(\lambda>0,\ \lambda\notin\mathbf{Z}).$$

$$c_{\frac{1}{2},\lambda,m}=\frac{(-\lambda)_{m+\frac{1}{2}}}{(\lambda)_{m+\frac{1}{2}}},\quad m+\tfrac{1}{2}\in\{0,1,2,\ldots\}.$$

No fixed sign.

(c) $$\pi^{+}_{\xi,\lambda}\text{ and }\pi^{-}_{\xi,\lambda}(\lambda+\xi\in\mathbf{Z}+\tfrac{1}{2},\ \lambda>0).$$

$$c_{\xi,\lambda,m}=\frac{(|m|-(\lambda+\frac{1}{2}))!}{(2\lambda+1)_{|m|-(\lambda+\frac{1}{2})}},\quad m\in\mathbf{Z}+\xi,\ |m|\geqslant\lambda+\tfrac{1}{2}.$$

Fixed sign.

(d) $$\pi^{0}_{\xi,\lambda}(\lambda+\xi\in\mathbf{Z}+\tfrac{1}{2},\ \lambda>0).$$

$$c_{\xi,\lambda,m}=\frac{(-1)^{m-\xi}}{(\lambda-\frac{1}{2}+m)!(\lambda+\frac{1}{2}-m)!},\quad m\in\left\{-\lambda+\tfrac{1}{2},-\lambda+\tfrac{3}{2},\ldots,\lambda-\tfrac{1}{2}\right\}.$$

No fixed sign except if $\lambda=\frac{1}{2}$, $\xi=0$.

Combining these results with Theorems 3.4, 4.7 and 5.4 and Prop. 4.2 we reobtain BARGMANN's [2] classiffication of all irreducible unitary representations of $SU(1,1)$:

THEOREM 6.2. *Any irreducible unitary representation of* $SU(1,1)$ *is unitarily equivalent to one and only one of the following representations:*

1) $\pi_{\xi,i\nu}(\xi=0,\frac{1}{2},\ \nu>0)$, $\pi_{0,0}$, $\pi^{+}_{\frac{1}{2},0}$, $\pi^{-}_{\frac{1}{2},0}$ *(unitary principal series).*

2) $\pi_{0,\lambda}(0<\lambda<\frac{1}{2})$ on Cl Span$\{\ldots,\phi_{-1},\phi_0,\phi_1,\ldots\}$

with respect to the inner product

$$<\phi_m\,\phi_n>:=\frac{(-\lambda+\frac{1}{2})_{|m|}}{(\lambda+\frac{1}{2})_{|m|}}\,\delta_{m,n}\ \textit{(complementary series)}.$$

3) $$\pi^{+}_{\xi,\lambda}\textit{ and }\pi^{-}_{\xi,\lambda}\left(\xi=0\text{ or }\tfrac{1}{2},\ \lambda=\xi+\tfrac{1}{2},\xi+\frac{3}{2},\ldots\right)$$

on

$$Cl\ \mathrm{Span}\{\phi_{\lambda+\frac{1}{2}}, \phi_{\lambda+3/2}, \ldots\}$$

and

$$Cl\ \mathrm{Span}\{\ldots, \phi_{-\lambda-3/2}, \phi_{-\lambda-\frac{1}{2}}\},$$

respectively, with respect to the inner product

$$<\phi_m, \phi_n> := \frac{(|m| - (\lambda+\frac{1}{2}))!}{(2\lambda+1)_{|m|-(\lambda+\frac{1}{2})}} \delta_{m,n} \text{ (discrete series).}$$

4) $\pi^0_{0,\frac{1}{2}}$ *(identity representation).*

6.3. Notes

6.3.1. Following BARGMANN [2], most authors prove Theorem 6.2 by infinitesimal methods. VILENKIN [43, Ch. VI] uses the method of the present paper. TAKAHASHI [39, §6] decides about unitarizability by considering whether $\pi_{\xi,\lambda,n,n}$ is a positive definite function on G.

6.3.2. A method related to this section was used in FLENSTED-JENSEN & KOORNWINDER [15] in order to find all irreducible unitary spherical representations of non-compact semisimple Lie groups G of rank one. They examined the nonnegativity of the coefficients in the addition formula for the spherical functions on G. See also [27, §6.4].

6.3.3. A generalization of Theorem 6.1 can be formulated for not necessarily abelian K and, partly, for K-finite τ, cf. [27, Theorems 6.4, 6.5].

REFERENCES

[1] ACZEL, J. *Lectures on functional equations and their applications*. Academic Press, New York, 1966.

[2] BARGMANN, V. Irreducible unitary representations of the Lorentz group. *Ann. of Math. 48* (1947), 568-640.

[3] BARUT, A. O. and E. C. PHILLIPS. Matrix elements of representations of non-compact groups in a continuous basis. *Comm. Math. Phys. 8* (1968), 52-65.

[4] BOERNER, H. *Representations of groups*. North-Holland Publishing Co., Amsterdam, 1970.

[B] BRUHAT, F. Sur les représentations induites des groupes de Lie. *Bull. Soc. Math. France 84* (1956), 97-205.

[6] DEANS, S. R. A unified Radon inversion formula. *J. Math. Phys. 19* (1978), 2346-2349.

[7] —— Gegenbauer transforms via the Radon transform. *SIAM J. Math. Anal. 10* (1979), 577-585.

[8] DIXMIER, J. Sur les représentations de certains groupes orthogonaux. *C. R. Acad. Sci. Paris 250* (1960), 3263-3265.

[9] DIJK, G. VAN. The irreducible unitary representations of $SL(2, \mathbf{R})$. Ch. XIII in "*Representations of locally compact groups with applications*" (T. H. Koornwinder, ed.), MC Syllabus 38, Math. Centrum, Amsterdam, 1979.

[10] ERDÉLYI, A., W. MAGNUS, F. OBERHETTINGER and F. G. TRICOMI. *Higher transcendental functions*, Vol. I. McGraw-Hill, New York, 1953.

[11] —— *Tables of integral transforms*. Vol. II. McGraw-Hill, New York, 1954.

[12] FARAUT, J. *Analyse harmonique sur les espaces Riemanniens symétriques de rang un*. Ecole d'été « Analyse harmonique », CIMPA, Université de Nancy I, 1980.

[17] FELL, J. M. G. Non-unitary dual spaces of groups. *Acta Math. 114* (1965), 267-310.

[14] FLENSTED-JENSEN, M. Paley-Wiener type theorems for a differential operator connected with symmetric spaces. *Ark. Mat. 10* (1972), 143-162.

[15] —— and T. H. KOORNWINDER. Positive definite spherical functions on a non-compact, rank one symmetric space. pp. 249-282 in "*Analyse harmonique sur les groupes de Lie, II*" (P. Eymard, J. Faraut, G. Schiffmann and R. Takahashi, eds.), Lecture Notes in Math. 739, Springer-Verlag, Berlin, 1979.

[16] GANGOLLI, R. On the Plancherel formula and the Paley-Wiener theorem for spherical functions on semisimple Lie groups. *Ann. of Math. (2) 93* (1971), 150-165.

[17] GELFAND, I. M., M. I. GRAEV and N. J. VILENKIN. *Generalized functions*, Vol. 5, *Integral geometry and its connection to the theory of group representations*. Academic Press, New York, 1968.

[18] GELFAND, I. M. and M. A. NAIMARK. Unitary representations of the Lorentz group (in Russian). *Izv. Akad. Nauk. SSSR, Ser. Mat., 11* (1947), 411-504.

[19] GODEMENT, R. A theory of spherical functions, I. *Trans. Amer. Math. Soc. 73* (1952), 496-556.

[20] —— Introduction aux travaux de A. Selberg. *Séminaire Bourbaki*, Exposé 144, Paris, 1957.

[21] HARISH-CHANDRA. Representations of a semisimple Lie group on a Banach space, I. *Trans. Amer. Math. Soc. 75* (1953), 185-243.

[22] —— Representations of semisimple Lie groups, II. *Trans. Amer. Math. Soc. 76* (1954), 26-65.

[23] —— Spherical functions on a semi-simple Lie group, I. *Amer. J. Math. 80* (1958), 241-310.

[24] —— Spherical functions on a semisimple Lie group, II. *Amer. J. Math. 80* (1958), 553-613.

[25] HELGASON, S. *Differential geometry and symmetric spaces.* Academic Press, New York, 1962.

[26] KOORNWINDER, T. H. A new proof of a Paley-Wiener type theorem for the Jacobi transform. *Ark. Mat. 13* (1975), 145-159.

[27] —— *The representation theory of* $SL(2, \mathbf{R})$, *a global approach.* Math. Centrum Report ZW 145, Amsterdam, 1980.

[28] KOSTERS, M. T. *A study of the representations of* $SL(2, \mathbf{C})$ *using non-infinitesimal methods.* Math. Centrum Report TW 190, Amsterdam, 1979.

[29] LEPOWSKY, J. Algebraic results on representations of semisimple Lie groups. *Trans. Amer. Math. Soc. 176* (1973), 1-57.

[30] MATSUSHITA, O. The Plancherel formula for the universal covering group of $SL(2, \mathbf{R})$. *Sc. Papers College of Gen. Ed. Univ. Tokyo 29* (1979), 105-123.

[31] MAUTNER, F. I. Review of reference [22]. *Math. Reviews 15* (1954), p. 398.

[32] MILLER, W., Jr. *Lie theory and special functions.* Academic Press, New York, 1968.

[33] NAIMARK, M. A. On the linear representations of the proper Lorentz group, (in Russian). *Dokl. Akad. Nauk. (N.S.) 97* (1954), 969-972.

[34] —— *Les représentations linéaires du groupe de Lorentz.* Dunod, Paris, 1962 (French translation of Russian edition, Moscow, 1958).

[35] POULSEN, N. S. On C^∞-vectors and intertwining bilinear forms for representations of Lie groups. *J. Functional Anal. 9* (1972), 87-120.

[36] SCHMID, W. Representations of semisimple Lie groups. pp. 185-235, in: *Representation theory of Lie groups*, (M. Atiyah e.a.), Cambridge University Press, Cambridge, 1979.

[37] SCHWARZ, G. W. Smooth functions invariant under the action of a compact Lie group. *Topology 14* (1975), 63-68.

[38] SPRINKHUIZEN-KUYPER, I. G. A fractional integral operator corresponding to negative powers of a certain second order differential operator. *J. Math. Anal. Appl. 72* (1979), 674-702.

[39] TAKAHASHI, R. Sur les fonctions sphériques et la formule de Plancherel dans le groupe hyperbolique. *Japan. J. Math. 31* (1961), 55-90.

[40] —— Sur les représentations unitaires des groupes de Lorentz généralisés. *Bull. Soc. Math. France 91* (1963), 289-433.

[41] —— Quelques résultats sur l'analyse harmonique dans l'espace symétrique non-compact de rang 1 du type exceptionnel. pp. 511-567, in: "*Analyse harmonique sur les groupes de Lie, II*" (P. Eymard, J. Faraut, G. Schiffmann and R. Takahashi, eds.). Lecture Notes in Math. 739, Springer-Verlag, Berlin, 1979.

[42] —— $SL(2, \mathbf{R})$. Ecole d'été, *Analyse harmonique.* CIMPA, Université de Nancy I, 1980.

[43] VILENKIN, N. J. *Special functions and the theory of group representations.* Amer. Math. Soc. Transl. of Math. Monographs, Vol. 22, Providence, R.I., 1968. (English translation of Russian edition, Moscow, 1965).

[44] WALLACH, N. R. Cyclic vectors and irreducibility for principal series representations, II. *Trans. Amer. Math. Soc. 164* (1972), 389-396.

[45] —— *Harmonic analysis on homogeneous spaces.* Marcel Dekker, New York, 1973.

[46] —— On the Selberg trace formula in the case of compact quotient. *Bull. Amer. Math. Soc. 82* (1976), 171-195.

[47] —— Representations of semisimple Lie groups and Lie algebras. pp. 154-246, in: *Lie theories and their applications*, (A. J. Coleman and P. Ribenboim, eds.). Queen's Papers in Pure and Applied Math., Queen's University, Kingston, Ontario, 1978.

[48] WARNER, G. *Harmonic analysis on semi-simple Lie groups*, Vol. I. Springer-Verlag, Berlin, 1972.
[49] —— *Harmonic analysis on semi-simple Lie groups*, Vol. II. Springer-Verlag, Berlin, 1972.
[50] WHITNEY, H. Differentiable even functions. *Duke Math. J. 10* (1943), 159-160.
[51] ZELOBENKO, D. P. and M. A. NAIMARK. A characterization of completely irreducible representations of a semisimple complex Lie group. *Soviet Math. Dokl. 7* (1966), 1403-1406.

(Reçu le 6 octobre 1981)

Tom H. Koornwinder
Mathematisch Centrum
P.O. Box 4079
1009 AB Amsterdam
The Netherlands