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THE REPRESENTATION THEORY OF $SL(2, \mathbf{R})$, A NON-INFINITESIMAL APPROACH

by Tom H. KOORNWINDER

ABSTRACT

The representation theory of $SL(2, \mathbf{R})$ is developed by the use of non-infinitesimal methods. This approach is based on an explicit knowledge of the matrix elements of the principal series with respect to the K-basis. The irreducible subquotient representations of the principal series are determined, and also their Naimark equivalences and unitarizability. All irreducible K-unitary, K-finite representations of $SL(2, \mathbf{R})$ are classified, where an inversion formula for the generalized Abel transform provides an important tool.

1. INTRODUCTION

In 1947 two papers appeared on the representation theory of the two prototypes of noncompact semisimple Lie groups, namely by BARGMANN [2] on $SL(2, \mathbf{R})$ and by GELFAND & NAIMARK [18] on $SL(2, \mathbf{C})$. The methods in the two papers are surprisingly different. Bargmann uses the infinitesimal (i.e. Lie algebraic) approach, while Gelfand & Naimark prefer non-infinitesimal (global) methods. In subsequent work to generalize these results for arbitrary noncompact semisimple Lie groups, the Bargmann approach has proved to be most successful, in particular by the work of Harish-Chandra. (However, it is interesting to note MAUTNER's [31] review of HARISH-CHANDRA's paper [22].)

Without denying the success of the infinitesimal approach, I want to add some motivation for a paper which favours the global approach:

(a) *The didactic argument.* The global approach is a more natural and direct one and it does not require so much sophisticated functional analysis as the infinitesimal approach.

(b) *Spin off to the theory of special functions and related harmonic analysis.* The global approach requires explicit knowledge of canonical matrix elements of representations as special functions. This provides new group-theoretic interpretations of well-known special functions and it also yields new interesting special functions.

(c) *The philosophical argument.* The representation theory of semisimple Lie groups is one of the great topics in mathematics at the moment. It is good to have several distinct philosophies existing beside each other for the development of this theory, where each philosophy provides a different insight.

In this paper a global approach to the representation theory of $SL(2, \mathbf{R})$ is presented. It is based on an explicit knowledge of the matrix elements of the principal series representations with respect to a basis which behaves nicely under the action of a maximal compact subgroup K.

Our program consists of four parts:

(i) Determine all irreducible subquotient representations of the principal series representations of $SL(2, \mathbf{R})$.

(ii) Determine which equivalence do exist between the representations in (i).

(iii) Prove that each irreducible representation of $SL(2, \mathbf{R})$ is equivalent to some representation in (i).

(iv) Which of the representations in (i) are unitarizable?

We will not only consider unitary representations, but, more generally, strongly continuous representations on a Hilbert space which are K-unitary and K-finite (cf. §2.1). Accordingly, we need a more general (but still non-infinitesimal) notion of equivalence than the notion of unitary equivalence, namely Naimark equivalence (cf. §4.1).

The four parts of the above program will be treated in Sections 3, 4, 5 and 6, respectively. We start in Section 2 with the conputation of the canonical matrix elements of the principal series representations. They can be expressed in terms of hypergeometric or, more elegantly, Jacobi functions. These explicit expressions will be used throughout the paper. Each section ends with extensive bibliographic notes.

The theory required for parts (i), (ii) and (iv) of our program can be developed in the more general situation of Hilbert representations of a locally compact group G which are multiplicity free with respect to a compact subgroup K, cf. the author's report [27]. This would make the theory applicable to $SO_0(n, 1)$ and

$SU(n, 1)$. For convenience, in order to avoid matrix manipulations, we restrict ourselves here to the case that the compact subgroup K is abelian.

The results of this paper may be generalized rather easily to the universal covering group of $SL(2, \mathbf{R})$. The extension to $SL(2, \mathbf{C})$ was done by KOSTERS [28], see also NAIMARK [34, ch. 3, §9]. Hopefully, an extension to $SO_0(n, 1)$ and $SU(n, 1)$ is feasible.

The reader of this paper is supposed to already have a modest knowledge about certain elements of semisimple Lie theory, like principal series and spherical functions. Suitable references will be given. Some of this preliminary material can also be found in the earlier version [27]. Modern accounts of the infinitesimal approach to $SL(2, \mathbf{R})$ can be found, for instance, in SCHMID [36, §2] or VAN DIJK [9]. TAKAHASHI [42] also presented a global approach to $SL(2, \mathbf{R})$, partly based on an earlier version of the present paper, partly (the global proof of Theorem 5.4) independently.

Finally, I would like to thank G. van Dijk and M. Flensted-Jensen for useful comments.

2. THE CANONICAL MATRIX ELEMENTS OF THE PRINCIPAL SERIES

2.1. PRELIMINARIES

Let G be a locally compact group satisfying the second axiom of countability (lcsc. group). A *Hilbert representation* of G is a strongly continuous but not necessarily unitary representation τ of G on some Hilbert space $\mathscr{H}(\tau)$ (which is always assumed to be separable). Let K be a compact subgroup of G. A Hilbert representation τ of G is called *K-unitary* if the restriction $\tau|_K$ of τ to K is a unitary representation of K. A Hilbert representation τ of G is called *K-finite* respectively *K-multiplicity free* if τ is K-unitary and each $\delta \in \hat{K}$ has finite multiplicity respectively multiplicity 1 or 0 in $\tau|_K$. If τ is K-multiplicity free then the *K-content* $\mathscr{M}(\tau)$ of τ is the set of all $\delta \in \hat{K}$ which have multiplicity 1 in $\tau|_K$.

Let K be a compact abelian subgroup of G and let τ be a K-multiplicity free representation of G. Choose an orthogonal basis $\{\phi_\delta \mid \delta \in \mathscr{M}(\tau)\}$ of $\mathscr{H}(\tau)$ such that

$$\tau(k)\phi_\delta = \delta(k)\phi_\delta, \qquad \delta \in \mathscr{M}(\tau), k \in K.$$

We call $\{\phi_\delta\}$ a *K-basis* for $\mathscr{H}(\tau)$ and the functions $\tau_{\gamma\delta}(\gamma, \delta \in \mathscr{M}(\tau))$, defined by

$$\tau_{\gamma,\delta}(g) := (\tau(g)\phi_\delta, \phi_\gamma), \quad g \in G, \tag{2.1}$$

the *canonical matrix elements* of τ (with respect to K).