2.4. Notes

Substitution of (2.23) and (2.22) yields $(m \geqslant n)$:

$$\begin{aligned}\pi_{\xi,\lambda,m,n}(a_t) \\ &= \frac{(\lambda+n+\frac{1}{2})_{m-n}}{(m-n)\,!}\,(sh\tfrac{1}{2}t)^{m-n}(ch\tfrac{1}{2}t)^{-m-n}\phi_{2i\lambda}^{(m-n,\,-m-n)}(\tfrac{1}{2}t) \\ &= \frac{(\lambda+n+\frac{1}{2})_{m-n}}{(m-n)\,!}\,(sh\tfrac{1}{2}t)^{m-n}(ch\tfrac{1}{2}t)^{m+n}\phi_{2i\lambda}^{(m-n,\,m+n)}(\tfrac{1}{2}t)\,.\end{aligned} \tag{2.28}$$

Application of (2.16) gives a similar result in the case $m < n$. Finally we conclude:

THEOREM 2.1. *The canonical matrix elements* $\pi_{\xi,\lambda,m,n}(a_t)$ $(\lambda \in \mathbf{C};\ \xi = 0$ or $\frac{1}{2};\ m, n \in \mathbf{Z} + \xi;\ t \in \mathbf{R})$ *of* $SU(1, 1)$ *can be expressed in terms of Jacobi functions by*

$$\pi_{\xi,\lambda,m,n}(a_t) = \frac{c_{\xi,\lambda,m,n}}{(|m-n|)\,!}\,(sh\tfrac{1}{2}t)^{|m-n|}\,(ch\tfrac{1}{2}t)^{m+n}\phi_{2i\lambda}^{(|m-n|,\,m+n)}(\tfrac{1}{2}t)\,, \tag{2.29}$$

where

$$c_{\xi,\lambda,m,n} := \begin{cases}(\lambda+n+\frac{1}{2})_{m-n} & \text{if } m \geqslant n\,, \\ (\lambda-n+\frac{1}{2})_{n-m} & \text{if } n \geqslant m\,.\end{cases} \tag{2.30}$$

In view of (2.24), formulas (2.29) and (2.30) describe the asymptotics of $\pi_{\xi,\lambda,m,n}$ near $t = 0$.

2.4. NOTES

2.4.1. The principal series of representations was first written down for $SL(2, \mathbf{R})$ by BARGMANN [2], for $SL(2, \mathbf{C})$ by GELFAND & NAIMARK [18], and for a general noncompact semisimple Lie group by HARISH-CHANDRA [21, §12].

2.4.2. BARGMANN [2, §10] already obtained explicit expressions in terms of hypergeometric functions for the canonical matrix elements of the irreducible unitary representations of $SL(2, \mathbf{R})$. He solved the differential equation satisfied by these matrix elements, which is obtained from the Casimir operator. VILENKIN [43, Ch. VI, §3] gives a derivation of these expressions which is similar to our derivation in §2.4, starting from the integral representation (2.15).

2.4.3. It follows from the present paper that the spherical functions for $SL(2, \mathbf{R})$ can be expressed as Jacobi functions of order $(\alpha, \beta) = (0, 0)$. More generally, the spherical functions on any noncompact real semisimple Lie group of rank 1 (i.e., $\dim(A) = 1$) can be written as Jacobi functions of certain order (cf. HARISH-CHANDRA [23, §13]). This motivated FLENSTED-JENSEN [14] to study harmonic analysis for Jacobi function expansions of quite general order (α, β), $\alpha \geqslant \beta \geqslant -\frac{1}{2}$. This research was continued in several papers by Flensted-Jensen and the author.

3. THE IRREDUCIBLE SUBQUOTIENT REPRESENTATIONS OF THE PRINCIPAL SERIES

3.1. SUBQUOTIENT REPRESENTATIONS

We start with the definition and some general properties and next derive an irreducibility criterium (Theorem 3.2) and a decomposition theorem 3.3.

Let G be a lcsc. group and let τ be a Hilbert representation of G. Let $\mathscr{H}_0$ be a closed subspace of $\mathscr{H}(\tau)$ and let P_0 be the orthogonal projection from $\mathscr{H}(\tau)$ onto $\mathscr{H}_0$. Define

$$\tau_0(g)v := P_0\tau(g)v\,, \quad g \in G, v \in \mathscr{H}_0\,. \tag{3.1}$$

Then $\tau(g) \in \mathscr{L}(\mathscr{H}_0)$ for each $g \in G$, $\tau_0(e) = id.$, and $g \to \tau_0(g)v : G \to \mathscr{H}_0$ is continuous for each $v \in \mathscr{H}_0$. If also

$$\tau_0(g_1 g_2) = \tau_0(g_1)\tau_0(g_2)\,, \quad g_1, g_2 \in G\,, \tag{3.2}$$

then τ_0 is a Hilbert representation of G on $\mathscr{H}_0$ and it is called a *subquotient representation* of τ. Formula (3.2) is clearly valid if $\mathscr{H}_0$ is an *invariant subspace* of $\mathscr{H}(\tau)$, i.e., if $\tau(g)v \in \mathscr{H}_0$ for all $g \in G$, $v \in \mathscr{H}_0$. In that case, τ_0 is called a *subrepresentation* of τ.

LEMMA 3.1. *Let $\mathscr{H}_0$ be a closed subspace of $\mathscr{H}(\tau)$, let $\mathscr{H}_2$ be the closed G-invariant subspace of $\mathscr{H}(\tau)$ which is generated by $\mathscr{H}_0$ and let $\mathscr{H}_1 := \mathscr{H}_2 \cap \mathscr{H}_0^{\perp}$. Then τ_0 is a subquotient representation if and only if $\mathscr{H}_1$ is G-invariant.*