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Artikel: THE REPRESENTATION THEORY OF SL(2, R), A NON-INFINITESIMAL APPROACH

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respectively $\beta \in \mathcal{M}(\sigma)$ the Fourier coefficient at the left respectively right hand side does not vanish identically in g_1, g_2. Hence $\mathcal{M}(\sigma) = \mathcal{M}(\tau)$ and

$$\tau_{\gamma,\beta}(g_1)\tau_{\beta,\delta}(g_2) = C_{\gamma,\delta}\sigma_{\gamma,\beta}(g_1)\sigma_{\beta,\delta}(g_2)\,.$$

This implies

$$\tau_{\gamma,\beta} = C_{\gamma,\beta}\sigma_{\gamma,\beta} \text{ and } \tau_{\beta,\delta} = C_{\beta,\delta}\sigma_{\beta,\delta} \text{ with } C_{\gamma,\beta}C_{\beta,\delta} = C_{\gamma,\delta}\,.$$

By repeating this argument we prove that $\tau_{\alpha,\beta} = C_{\alpha,\beta}\sigma_{\alpha,\beta}$ for all $\alpha, \beta \in \mathcal{M}(\sigma)$ and that $C_{\alpha,\beta}C_{\beta,\delta} = C_{\alpha,\delta}$, i.e. $C_{\alpha,\beta} = C_{\alpha,\delta}/C_{\beta,\delta}$. □

COROLLARY 4.6. *Let G be an lcsc. group with compact abelian subgroup K. Then Naimark relatedness is an equivalence relation in the class of K-multiplicity free representations of G.*

4.2. THE CASE $SU(1, 1)$

Consider irreducible subquotient representations of $\pi_{\xi,\lambda}$ as classified in Theorem 3.4. By comparing K-contents it follows that the only possible nontrivial Naimark equivalences are:

$$\pi_{\xi,\lambda} \simeq \pi_{\xi,\mu}(\lambda+\xi,\ \mu+\xi \notin \mathbf{Z}+\tfrac{1}{2},\ \lambda \neq \mu)$$

and

$$\pi^{+}_{\xi,\lambda} \simeq \pi^{+}_{\xi,-\lambda},\quad \pi^{0}_{\xi,\lambda} \simeq \pi^{0}_{\xi,-\lambda},\quad \pi^{-}_{\xi,\lambda} \simeq \pi^{-}_{\xi,-\lambda}$$
$$(\lambda+\xi \in \mathbf{Z}+\tfrac{1}{2},\ \lambda \neq 0)\,.$$

Suppose that σ and τ are irreducible subquotient representations of $\pi_{\xi,\lambda}$ and $\pi_{\xi,\mu}$, respectively, and that $\phi_m \in \mathcal{H}(\sigma) \cap \mathcal{H}(\tau)$ for some $m \in \mathbf{Z} + \xi$. It follows from Theorem 4.5 that $\sigma \simeq \tau$ iff $\tau_{\xi,\lambda,m,m} = \pi_{\xi,\mu,m,m}$. This last identity already holds if it is valid for the restrictions to A. In view of (2.29) and (2.30) we have: $\sigma \simeq \tau$ iff

$$\phi^{(0,\,2m)}_{2i\lambda}(t) = \phi^{(0,\,2m)}_{2i\mu}(t)\,, \qquad t \in \mathbf{R}\,. \tag{4.5}$$

Formula (4.5) holds if $\lambda = \pm\mu$ (cf. (2.26)). Conversely, assume (4.5) and expand both sides of (4.5) as a power series in $-(sh\ t)^2$ by using (2.23) and (2.20). The coefficients of $-(sh\ t)^2$ yield the equality

$$(m+1+\lambda)(m+1-\lambda) = (m+1+\mu)(m+1-\mu)$$

Hence $\lambda = \pm\mu$. We have proved:

THEOREM 4.7. *Let* σ *and* $\tau(\sigma \neq \tau)$ *be irreducible subquotient representations of the principal series. Then* σ *is Naimark equivalent to* τ *in precisely the following situations (cf. the notation of Theorem 3.4):*

(a) $$\pi_{\xi,\lambda} \simeq \pi_{\xi,-\lambda} (\lambda+\xi \notin \mathbf{Z}+\tfrac{1}{2}, \lambda \neq 0)$$

(b) $$\pi^{+}_{\xi,\lambda} \simeq \pi^{+}_{\xi,-\lambda}, \pi^{0}_{\xi,\lambda} \simeq \pi^{0}_{\xi,-\lambda}, \pi^{-}_{\xi,\lambda} \simeq \pi^{-}_{\xi,-\lambda} \ (\lambda+\xi \in \mathbf{Z}+\tfrac{1}{2}, \lambda \neq 0).$$

Remark 4.8. It follows from Theorem 3.4 and Theorem 4.7 that each irreducible subquotient representation of some $\pi_{\xi,\lambda}$ is Naimark equivalent to some irreducible subrepresentation of some $\pi_{\xi,\lambda}$.

It follows from Theorems 4.7 and 4.5 that for each $\xi \in \{0, \frac{1}{2}\}$ and $\lambda \in \mathbf{C}\backslash\{0\}$ we have identities

(4.6) $$\pi_{\xi,-\lambda,m,n} = C_{\xi,\lambda,m,n}\, \pi_{\xi,\lambda,m,n}$$

for certain nonzero complex constants $C_{\xi,\lambda,m,n}$, where $m, n \in \mathbf{Z} + \xi$ and, if $\lambda + \xi \in \mathbf{Z} + \frac{1}{2}$, we have the further restriction that $m, n \in (-\infty, -|\lambda| - \frac{1}{2}]$ or $m, n \in [-|\lambda| + \frac{1}{2}, |\lambda| - \frac{1}{2}]$ or $m, n \in [|\lambda| + \frac{1}{2}, \infty)$. Indeed, it follows from (2.29) and (2.26) that (4.6) holds with

(4.7) $$C_{\xi,\lambda,m,n} = \frac{c_{\xi,-\lambda,m,n}}{c_{\xi,\lambda,m,n}}.$$

A calculation using (4.7) and (2.30) shows that

(4.8) $$C_{\xi,\lambda,m,n} = c_{\xi,\lambda,m}/c_{\xi,\lambda,n}$$

with

(4.9) $$\begin{aligned} c_{\xi,\lambda,m} &= \text{const.}\,\frac{\Gamma(-\lambda+m+\frac{1}{2})}{\Gamma(\lambda+m+\frac{1}{2})} = \text{const.}\,\frac{\Gamma(-\lambda-m+\frac{1}{2})}{\Gamma(\lambda-m+\frac{1}{2})} \\ &= \text{const.}\,(-1)^{m-\xi}\,\Gamma(-\lambda+m+\tfrac{1}{2})\,\Gamma(-\lambda-m+\tfrac{1}{2}) \\ &= \text{const.}\,\frac{(-1)^{m-\xi}}{\Gamma(\lambda+m+\frac{1}{2})\,\Gamma(\lambda-m+\frac{1}{2})}. \end{aligned}$$

If $\lambda + \xi \notin \mathbf{Z} + \frac{1}{2}$ then we can use all alternatives for $c_{\xi,\lambda,m}$, but if $\lambda + \xi \in \mathbf{Z} + \frac{1}{2}$ then we can use precisely one alternative. Now, by Theorem 4.5, we obtain:

PROPOSITION 4.9. *Let* $\sigma \overset{A}{\simeq} \tau$ *be one of the equivalences of Theorem 4.7 with* σ *being a subquotient representation of* $\pi_{\xi,\lambda}$. *Then*

(4.10) $$A\phi_m = c_{\xi,\lambda,m}\,\phi_m,$$

where $m \in \mathbf{Z} + \xi$ *such that* $\delta_m \in \mathcal{M}(\sigma)$ *and* $c_{\xi,\lambda,m}$ *is given by (4.9).*

4.3. NOTES

4.3.1. Definition 4.1 of Naimark relatedness goes back to NAIMARK [33]. He introduced this concept in the context of representations of the Lorentz group on a reflexive Banach space. Next he gave a much more involved definition in his book [34, Ch. 3, §9, No. 3]. Afterwards, many different versions of this definition appeared in literature, which all refer to [34]. We mention ZELOBENKO & NAIMARK [51, Def. 2] ("weak equivalence" for representations on locally convex spaces), FELL [13, §6] (Naimark relatedness for "linear system representations") and WARNER [48, p. 232 and p. 242]. Warner starts with the definition of Naimark relatedness for Banach representations of an associative algebra over **C** (this definition is similar to our Definition 4.1) and next he defines Naimark relatedness for Banach representations of an lcsc. group G in terms of Naimark relatedness for the corresponding representations of $M_c(G)$ or (equivalently) $C_c(G)$. Warner's definition seems to be standard now. POULSEN [35, Def. 33] gives Naimark's original definition [33] and he calls it weak equivalence. FELL [13] (see also WARNER [48, Theorem 4.5.5.2]) proved that, for K-finite Banach representations of a connected unimodular Lie group, two representations are Naimark related iff they are infinitesimally equivalent.

4.3.2. Our implication (*c*) ⇒ (*a*) in Theorem 4.5 is related to WALLACH [44, Cor. 2.1]. Theorem 4.7 can be formulated for general semisimple Lie groups G. If $\pi_{\xi,\lambda}$ is an irreducible principal series representation and if $s \in W$ then $\pi_{\xi,\lambda} \simeq \pi_{\xi^s,\, s\cdot\lambda}$ (cf. WALLACH [44, Theorem 3.1]). This yields part (a). Regarding part (b) see LEPOWSKY's [29, Theorem 9.8] result that $\pi_{\xi,\lambda}$ and $\pi_{\xi^s,\, s\cdot\lambda}$ have equivalent composition series.

4.3.3. Theorem 4.7 was first proved in the unitarizable cases by BARGMANN [2]. He used infinitesimal methods. TAKAHASHI [39] proved Theorem 4.7 (again in the unitarizable cases) by calculating the diagonal matrix elements $\pi_{\xi,\lambda,m,n}(a_t)$ and by observing that they are even in λ. GELFAND, GRAEV & VILENKIN [17, Ch. VII, §4] obtained Theorem 4.7 by working in the noncompact realization of the principal series and by explicitly constructing all possible intertwining operators.

4.3.4. Analogues of the results in §4.1 hold for nonabelian K and (in Lemmas 4.3, 4.4 and Corollary 4.6) for K-finite representations, cf. [27, §4].