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Artikel: THE REPRESENTATION THEORY OF SL(2, R), A NON-INFINITESIMAL APPROACH

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5.2. Spherical functions of type δ

Let G be a unimodular lcsc. group with compact subgroup K. Let

$$K^* := \{(k, k) \in G \times K \mid k \in K\}.$$

Let $\delta \in \hat{K}$ and let τ be a K-unitary representation of G. Then $\tau \otimes \check{\delta}$ ($\check{\delta}$ the contragredient representation to δ) is a K^*-unitary representation of $G \times K$ on $\mathscr{H}(\tau) \otimes \mathscr{H}(\delta)$.

Lemma 5.1. *The multiplicity of* δ *in* $\tau|_K$ *is equal to the multiplicity of the representation* 1 *of* K^* *in* $\tau \otimes \check{\delta}|_{K^*}$. τ *is irreducible iff* $\tau \otimes \check{\delta}$ *is irreducible.* τ *is unitary iff* $\tau \otimes \check{\delta}$ *is unitary.*

This can be proved immediately. By using the results summarized in §5.1 we conclude that $(G \times K, K^*)$ is a Gelfand pair if there exists a continuous involutive homomorphism α on G such that for each $(g, k) \in G \times K$ we have $\alpha(g) = k_1 g^{-1} k_2$, $\alpha(k) = k_1 k^{-1} k_2$ for certain $k_1, k_2 \in K$. Furthermore, if $(G \times K, K^*)$ is a Gelfand pair and if the irreducible representation τ of G is unitary or K-finite then τ is K-multiplicity free. In particular, this applies to $SU(1, 1)$:

Proposition 5.2. *If* $G = SU(1, 1)$ *then* $(G \times K, K^*)$ *is a Gelfand pair.*

Proof. For $g \in SU(1, 1)$ define $\alpha(g) := {}^t(g^{-1})$. Then α is a continuous involutive automorphism on G and $\alpha(a_t) = a_{-t}$ on A, $\alpha(u_\theta) = u_{-\theta}$ on K. Since $G = KAK$, α has the required properties. □

Let $(G \times K, K^*)$ be a Gelfand pair. Identify $G \times \{e\}$ with G. A spherical function on $G \times K$ is completely determined by its restriction to G. By using the results mentioned in §5.1 we obtain the following properties. First, a continuous function ϕ on G is the restriction to G of a spherical function on $G \times K$ iff $\phi \neq 0$ and

$$\phi(x)\phi(y) = \int_K \phi(xkyk^{-1})dk, \qquad x, y \in G.$$

Next, let

$$\begin{aligned} &I_c(G) \ (\text{or } I_c^\infty(G)) \\ &:= \{f \in C_c(G) \ (\text{or } C_c^\infty(G)) \mid f(kgk^{-1}) = f(g), \\ &\qquad g \in G, k \in K\}. \end{aligned}$$

These are commutative topological algebras under convolution and their characters are precisely of the form (5.1), where ϕ is a spherical function on $G \times K$. If ϕ is a spherical function on $G \times K$ then there is a $\delta \in \hat{K}$ such that for all $x \in G$ the function $k \to \phi(xk)$ on K belongs to δ. Then δ is called a *spherical function of type* δ on G (with respect to K), cf. GODEMENT [19]. It is funny that spherical functions of type δ are on the one hand generalizations of ordinary spherical functions for (G, K), on the other hand restrictions to G of ordinary spherical functions for $(G \times K, K^*)$.

For convenience, we take a one-dimensional $\delta \in \hat{K}$. Then a spherical function ϕ on $G \times K$ is of type δ iff

$$\phi(xk) = \phi(kx) = \delta(k)\phi(x), \qquad x \in G, k \in K .$$

Let

$$\begin{aligned} &I_{c,\delta}(G) \ \left(\text{or } I^{\infty}_{c,\delta}(G)\right) \\ &:= \left\{f \in C_c(G) \ \left(\text{or } C^{\infty}_c(G)\right) \mid f(xk) = f(kx) \right. \\ &\qquad \left. = \delta(k)f(x),\ x \in G,\ k \in K\right\} . \end{aligned}$$

These are closed subalgebras of $I_c(G)$ (or $I^{\infty}_c(G)$) and their characters are precisely of the form (5.1), where ϕ is a spherical function of type δ. Finally, if τ is a K-unitary representation of G and if $\mathscr{H}(\tau)$ contains a unit vector v satisfying $\tau(k)v = \delta(k)v$, unique up to a constant factor, then $x \to (\tau(x)v, v)$ is a spherical function of type δ.

5.3. THE GENERALIZED ABEL TRANSFORM

Let G be a connected noncompact real semisimple Lie group with finite center. Use the notation of §2.2. For given Haar measures dk, da, dn on K, A, N, respectively, normalize the Haar measure on G such that

$$(5.2) \qquad \int_G f(g)dg = \int_{K \times A \times N} f(kan)e^{2\rho(\log a)}\, dk\, da\, dn, \quad f \in C_c(G)$$

(cf. HELGASON [25, Ch. X, Prop. 1.11]). Note the property

$$(5.3) \qquad \int_N f(n)dn = e^{2\rho(\log a)} \int_N f(ana^{-1})dn, \quad f \in C_c(N),\ a \in A$$

(cf. [25, Ch. X, proof of Prop. 1.11]).