

5.3. The generalized Abel transform

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These are commutative topological algebras under convolution and their characters are precisely of the form (5.1), where ϕ is a spherical function on $G \times K$. If ϕ is a spherical function on $G \times K$ then there is a $\delta \in \hat{K}$ such that for all $x \in G$ the function $k \rightarrow \phi(xk)$ on K belongs to δ . Then δ is called a *spherical function of type δ* on G (with respect to K), cf. GODEMENT [19]. It is funny that spherical functions of type δ are on the one hand generalizations of ordinary spherical functions for (G, K) , on the other hand restrictions to G of ordinary spherical functions for $(G \times K, K^*)$.

For convenience, we take a one-dimensional $\delta \in \hat{K}$. Then a spherical function ϕ on $G \times K$ is of type δ iff

$$\phi(xk) = \phi(kx) = \delta(k)\phi(x), \quad x \in G, k \in K.$$

Let

$$\begin{aligned} & I_{c, \delta}(G) \text{ (or } I_{c, \delta}^\infty(G)) \\ & := \{f \in C_c(G) \text{ (or } C_c^\infty(G)) \mid f(xk) = f(kx) \\ & \quad = \delta(k)f(x), x \in G, k \in K\}. \end{aligned}$$

These are closed subalgebras of $I_c(G)$ (or $I_c^\infty(G)$) and their characters are precisely of the form (5.1), where ϕ is a spherical function of type δ . Finally, if τ is a K -unitary representation of G and if $\mathcal{H}(\tau)$ contains a unit vector v satisfying $\tau(k)v = \delta(k)v$, unique up to a constant factor, then $x \rightarrow (\tau(x)v, v)$ is a spherical function of type δ .

5.3. THE GENERALIZED ABEL TRANSFORM

Let G be a connected noncompact real semisimple Lie group with finite center. Use the notation of §2.2. For given Haar measures dk, da, dn on K, A, N , respectively, normalize the Haar measure on G such that

$$(5.2) \quad \int_G f(g)dg = \int_{K \times A \times N} f(kan)e^{2\rho(\log a)} dk da dn, f \in C_c(G)$$

(cf. HELGASON [25, Ch. X, Prop. 1.11]). Note the property

$$(5.3) \quad \int_N f(n)dn = e^{2\rho(\log a)} \int_N f(ana^{-1})dn, f \in C_c(N), a \in A$$

(cf. [25, Ch. X, proof of Prop. 1.11]).

For $\lambda \in \mathfrak{a}_\mathbb{C}^*$ let U^λ be the representation of G induced by the one-dimensional representation $an \rightarrow e^{\lambda(\log a)}$ of the subgroup AN :

$$(5.4) \quad (U^\lambda(g)f)(k) := e^{-(\rho+\lambda)H(g^{-1}k)} f(u(g^{-1}k)), \quad f \in L^2(K), g \in G, k \in K.$$

The representation U^λ is easily seen to split as a direct sum of principal series representations $\pi_{\xi, \lambda}$. U^λ restricted to K is the left regular representation of K .

Let $\delta \in \widehat{K}$. For convenience, suppose that δ is one-dimensional. The *generalized Abel transform* $f \rightarrow F_f^\delta: I_{c, \delta}(G) \rightarrow C_c(A)$ is defined by

$$(5.5) \quad F_f^\delta(a) := e^{\rho(\log a)} \int_N f(an) dn, \quad a \in A.$$

If $G = SU(1, 1)$ and $\delta = 1$ then this transform can be rewritten as the classical Abel transform, cf. §5.4.

PROPOSITION 5.3. *The mapping $f \rightarrow F_f^\delta$ is a continuous homomorphism (with respect to convolution on G and A , respectively) from $I_{c, \delta}^\infty(G)$ to $C_c^\infty(A)$. Furthermore,*

$$(5.6) \quad \int_A F_f^\delta(a) e^{-\lambda(\log a)} da = \int_G f(g) (U^\lambda(g^{-1})\check{\delta}, \check{\delta}) dg, \quad f \in I_{c, \delta}^\infty(G), \lambda \in \mathfrak{a}_\mathbb{C}^*,$$

where (\cdot, \cdot) denotes the inner product on $L^2(K)$.

Proof. The continuity is immediate. The homomorphism property follows easily from (5.2) and (5.3) (cf. WARNER [49, pp. 34, 35]). For the proof of (5.6) substitute (5.4) into the right hand side of (5.6):

$$\begin{aligned} \int_G f(g) (U^\lambda(g^{-1})\check{\delta}, \check{\delta}) dg &= \int_G \int_K f(g) e^{-(\rho+\lambda)H(gk)} \delta((u(gk))^{-1}k) dk dg \\ &= \int_G f(g) e^{-(\rho+\lambda)H(g)} \delta((u(g))^{-1}) dg \\ &= \int_{K \times A \times N} f(kan) e^{(\rho-\lambda)\log a} \delta(k^{-1}) dk da dn \\ &= \int_A \int_N f(an) e^{(\rho-\lambda)\log a} dn da \\ &= \int_A F_f^\delta(a) e^{-\lambda(\log a)} da. \end{aligned}$$

□

Now let $G = SU(1, 1)$. Write $F_f^n(t)$ and $I_{c, n}^\infty(G)$ instead of $F_f^{\delta_n}(a_t)$ and $I_{c, \delta_n}^\infty(G)$, respectively. If $n \in \mathbf{Z} + \xi$ then (5.5) and (5.6) take the form

$$(5.7) \quad F_f^n(t) = e^{\frac{1}{2}t} \int_{-\infty}^{\infty} f(a_t n_z) dz$$

and

$$(5.8) \quad \int_{-\infty}^{\infty} F_f^n(t) e^{-\lambda t} dt = \int_G f(g) \pi_{\xi, \lambda, n, n}(g^{-1}) dg, \quad f \in I_{c, n}^\infty(G), \lambda \in \mathbf{C},$$

where $dg = (2\pi)^{-1} e^t d\theta dt dz$ if $g = u_\theta a_t n_z$.

5.4. THE MAIN THEOREM

It is the purpose of this section to prove:

THEOREM 5.4. *Let τ be an irreducible K -unitary representation of $SU(1, 1)$ which is K -finite or unitary. Then τ is Naimark equivalent to an irreducible subrepresentation of some principal series representation $\pi_{\xi, \lambda}$.*

By Proposition 5.2 τ is K -multiplicity free. If $\delta_n \in \mathcal{M}(\tau)$ then write $\tau_{n, n}$ instead of $\tau_{\delta_n, \delta_n}$. In view of Theorem 4.5 and Remark 4.8 it is sufficient for the proof of Theorem 5.4 to show that for some $\delta_n \in \mathcal{M}(\tau)$, for some $\lambda \in \mathbf{C}$ and for $\xi \in \{0, \frac{1}{2}\}$ with $n \in \mathbf{Z} + \xi$ we have

$$(5.9) \quad \tau_{n, n} = \pi_{\xi, \lambda, n, n}.$$

Both sides of (5.9) are spherical functions of type δ_n . Then (5.9) holds if the corresponding characters on $I_{c, n}^\infty(G)$ are equal. Hence Theorem 5.4 will follow from

PROPOSITION 5.5. *Let $G = SU(1, 1)$, $n \in \frac{1}{2}\mathbf{Z}$. Let α be a continuous character on $I_{c, n}^\infty(G)$. Then*

$$(5.10) \quad \alpha(f) = \int_G f(g) \pi_{\xi, \lambda, n, n}(g^{-1}) dg, \quad f \in I_{c, n}^\infty(G),$$

for some $\lambda \in \mathbf{C}$ and for $\xi \in \{0, \frac{1}{2}\}$ such that $n \in \mathbf{Z} + \xi$.