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Since DAG is logspace complete in NSPACE (log n), it suffices to show that

 $DAG \in DSPACE (log n)/log \Rightarrow DAG \in DSPACE (log n)$.

Suppose that DAG = S: h, where $S \in DSPACE$ (log n) and

 $|h(n)| \leq k \log_2 n$.

Then, guided by the self-reducibility of DAG, we can test whether $(\Psi, s, t) \in DAG$ by performing the following computation for each string w of length $\leq k \log_2 n$:

```
v := s;
while v has out-degree 2 do
v := \text{if } w \cdot (\Psi, v_0, t) \in S \text{ then } v_0 \text{ else } v_1.
```

If v is ever set equal to t then accept (Ψ, s, t) ; otherwise, reject it. It is clear that this method recognizes DAG deterministically within space 0 (log n).

6. The Method of Recursive Definition

Let K be a subset of $\{0, 1\}^*$, and let $C_K : \{0, 1\}^* \to \{0, 1\}$ be the characteristic function of K. By a recursive definition of C_K we mean a rule that specifies C_K on a "basis set" $A \subseteq \{0, 1\}^*$, and uniquely determines C_K on the rest of $\{0, 1\}^*$ by a recurrence formula of the form

$$C_{K}(x) = F(x, C_{K}(f_{1}(x)), C_{K}(f_{2}(x)), ..., C_{K}(f_{t}(x))),$$
$$x \in \{0, 1\}^{*} - A.$$

Example 1. Let G be a game, as defined in Section 4, and let G be the set of positions from which the player to move can force a win. Then G is uniquely determined by

- (i) if $x \in W$ then $x \in G$
- (ii) if $x \in \{0, 1\}^*$ W then $x \in G \Leftrightarrow F_0(x) \notin G$ or $F_1(x) \notin G$.

Example 2. Let $(\langle A, G_0, G_1 \rangle)$ be a self-reducibility structure for the set $K \subseteq \{0, 1\}^*$. Then K is determined uniquely by its intersection with A, together with the recurrence

for $x \notin A$, $x \in K \Leftrightarrow G_0(x) \in K \cup G_1(x) \in K$.

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L'Enseignement mathém., t. XXVIII, fasc. 3-4.

The theme of the present section is that, when C_K has a simple enough recursive definition, bounds on the nonuniform complexity of K yield bounds on its uniform complexity. The idea is as follows. Suppose K = S : h, and C_K is determined by its values on A, together with the recurrence formula

$$C_{K}(x) = F(x, C_{K}(f_{1}(x)), ..., C_{K}(f_{t}(x))), x \in \{0, 1\}^{*} - A,$$

where

$$|f_1(x)| = |f_t(x)| = |x|.$$

For any string w, define $K_w = \{x \mid wx \in S\}$. Then, for $x \in A$, we can make the following assertion:

$$\begin{aligned} & x \in \mathbf{K} \Leftrightarrow \exists w \left[x \in \mathbf{K}_{w} \right] \land \forall y \left[C_{K_{w}}(y) \right. \\ & = F\left(y, C_{K_{w}}\left(f_{1}\left(y \right) \right), ..., C_{K_{w}}\left(f_{t}\left(y \right) \right) \right]. \end{aligned}$$

Here, w ranges over all strings of length |h(|x|)|, and y ranges over all strings of the same length as x. The above formula suggests a uniform algorithm to test membership in K by searching through all choices of w and y. Further, the quantifier structure of the formula allows us to conclude that K lies in $\sum_{n=2}^{p}$, provided that $|h(n)\rangle|$ is bounded by a polynomial in n, S is in P, and F is computable in polynomial time.

As an illustration of this approach, we prove that, if NP has small circuits, then $\bigcup_{i} \sum_{i}^{p} = \sum_{i}^{p}$, i.e., the polynomial-time hierarchy collapses. Originally we proved this with \sum_{i}^{p} replaced by \sum_{3}^{p} . The improvement is due to M. Sipser.

THEOREM 6.1. If
$$NP \subseteq P/poly$$
 then $\sum_{2}^{p} = \bigcup_{i=1}^{\infty} \sum_{j=1}^{p} i^{p}$.

The proof of this theorem requires the following lemma.

LEMMA 6.2. If
$$NP \subseteq P/poly$$
 then $\bigcup_{i=1}^{\infty} \sum_{i=1}^{p} \subseteq P/poly$.

Proof. Let E_i be the set of encodings of true sentences of the form

(*)
$$Q_1 \vec{x}_1 Q_2 \vec{x}_2 \dots Q_i \vec{x}_i F(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_i)$$

where $Q_1 = \exists$, the Q_j are alternately \exists and \forall , \vec{x}_j is shorthand for the triple $x_{j_1}, x_{j_2}, ..., x_{j,r_j}$ of Boolean variables, and F is a propositional formula. Let A_i be defined in the same way, except that $Q_1 = \forall$. It is known that E_i is logspace complete in $\sum_{i=1}^{p}$, and A_i is logspace complete in

 $\prod_{i=1}^{p} Also, it is clear that <math>A_i \in P/poly \Leftrightarrow E_i \in P/poly$. It suffices for the lemma to prove that $E_i \in P/poly$ for all *i*.

By hypothesis, $E_1 \in P/poly$. We proceed by induction on *i*. Assume $E_{i-1} \in P/poly$; then $A_{i-1} \in P/poly$. Thus there exists a set $S \in P$, a constant k and a function $h: N \to \{0, 1\}^*$ such that $|h(n)| \leq k + n^k$ and $x \in A_{i-1} \Leftrightarrow h(|x|) \cdot x \in S$.

If y is the encoding of a sentence of the form (*), and \vec{a} is a t_1 -tuple of boolean variables, let $y_{\vec{a}}$ denote the encoding of the sentence that results from y by deleting the quantifier Q_1 and substituting \vec{a} for \vec{x}_i in $F(\vec{x}_1, \vec{x}_2, ..., \vec{x}_i)$. We choose our encoding conventions and method of substitution so that the length of $y_{\vec{a}}$ is equal to the length of y.

Since $S \in P$, the following set T is in NP:

 $T = \{wy \mid \text{ for some } \vec{a}, w \cdot y_{\vec{a}} \in S\}.$

By hypothesis $T \in P/polv$, so there exist $S' \in P$, $k' \in N$ and $h' : N \to \{0, 1\}^*$ so that $|h'(n)| \leq k' + n^{k'}$ and $x \in T \Leftrightarrow h'(|x|) \cdot x \in S$. Then $y \in A_i \Leftrightarrow$ for some \vec{a} , $y_{\vec{a}} \in E_{i-1} \Leftrightarrow$ for some a,

$$h(|y_{a}^{\rightarrow}) \cdot y_{a}^{\rightarrow} \in \mathbf{S} \Leftrightarrow h(|y_{a}^{\rightarrow}|) \cdot y \in T \Leftrightarrow h'(|h(|y_{a}^{\rightarrow}|) \cdot y|(\cdot h(|y_{a}^{\rightarrow}|) \cdot y \in \mathbf{S}')$$

But the prefix $h'(|h(|y_a|) \cdot y|(\cdot h(|y_a|))$ is a polynomial-bounded function of |y|; also S' $\in P$. These two facts together establish that $A_i \in P/poly$.

Proof of Theorem 6.1. It suffices to prove that $NP \subseteq P/poly \Rightarrow \prod_{3}^{p} \subseteq \sum_{2}^{p}$; for this it is sufficient to prove that the set A_{3} is in \sum_{2}^{p} . Our proof is based on the fact that A_{3} has an easty-to-evaluate recursive definition of the form $C_{A_{3}}(y) = R(y, C_{A_{3}}(y'), C_{A_{3}}(y''))$. Consider a sentence y of the form

$$Q_1 x_1 Q_2 x_2 \dots Q_n x_n F(x_1, x_2, \dots, x_n)$$

where the string of quantifiers $Q_1 Q_2 \dots Q_n$ is contained in $\forall * \exists * \forall *$.

Let

 $y' = Q_2 x_2 \dots Q_n x_n F(0, x_2, \dots, x_n)$

and

$$y'' = Q_2 x_2 \dots Q_n x_n F(1, x_2, \dots, x_n)$$

Then

 $C_{A3}(y) = (\text{if } Q_1 = \forall \text{ then } C_{A3}(y') \land C_{A3}(y'') \text{ else } C_{A3}(y') \cup C_{A3}(y''))$. C_{A3} is uniquely determined by this recursive definition which is of the form $C_{A3}(y) = R((y, C_{A3}(y'), C_{A3}(y'')))$, coupled with its values on the "basis set" consisting of sentences without quantifiers. By Lemma 6.2, $A_3 \in P/poly$. Thus $A_3 = S:h$ where $S \in P$ and $|h(n)| \leq k + n^k$. For each $w \in \{0, 1\}^*$ define $f_w: \{0, 1\}^* \to \{0, 1\}$ by $f_w(x) = 1 \Leftrightarrow wx \in S$. Then membership of y in A_3 , in the case where y contains at least one quantifier, is expressed by the following formula:

(**)
$$\exists w \forall z [f_w(y) = 1 \land f_w(z) = R(z, f_w(z'), f_w(z''))]$$

Here w ranges over all strings of length $\leq k + |y|^k$, and z ranges over all strings of length |y|. Also, with the help of a polynomial-time algorithm to test membership in S, the property $f_w(y) = 1$ and

$$f_{w}(z) = R\left(z, f_{w}(z'), f_{w}(z'')\right)$$

can be tested in polynomial time. Thus the $\exists \forall$ form of (**) establishes that $A_3 \in \sum_{j=1}^{p} A_j$.

Theorem 6.1 has a number of corollaries.

COROLLARY 6.3. If R = NP then $\bigcup_{i} \sum_{i}^{p} = \sum_{i}^{p}$.

This follows immediately from the observation [1] that every set in R has small circuits.

The next corollary concerns sparse sets. A set S is sparse [6, 7] if

$$\exists \ c \ orall n \geqslant 2, \ \mid \mathrm{S} \cap \{0,1\}^n \mid \, \leqslant n^c$$
 .

COROLLARY 6.4. If there is a sparse set S that is complete in NP with respect to polynomial time Turing reducibility (cf. Cook [4]), then

$$\bigcup_{i} \sum_{i}^{p} = \sum_{i}^{p}.$$

This corollary follows immediately from Theorem 6.1 once it is noted that the existence of such an S implies that every set in NP has small circuits. Corollary 6.4 should be compared with results of Mahaney [11] and Fortune [6] which show that, if there exists a sparse or co-sparse set which is complete in NP with respect to many-one polynomial-time reducibility (Karp [8]) then P = NP. Note that Corollary 6.4 has a weaker conclusion than the results of Mahaney and Fortune, but also a weaker hypothesis.

Let ZEROS denote the following decision problem: given a prime q and a set $\{p_1(x), p_2(x), ..., p_n(x)\}$ of sparse polynomials with integer coefficients, to determine whether there exists an integer x such that, for $i = 1, 2, ..., n, p_i(x) \equiv 0 \mod q$.

COROLLARY 6.5. If $ZEROS \in P/poly$, then $\bigcup \sum_{i}^{p} = \sum_{i}^{p} 2^{i}$.

This is based on Plaisted's result [15] that every problem in NP can be solved in polynomial time with the help of an oracle for ZEROS together with a polynomial-bounded number of advice bits. Thus $NP \subseteq P/poly$ if $ZEROS \in P/poly$.

THEOREM 6.6. (Meyer) $EXPTIME \subseteq P/poly \Leftrightarrow EXPTIME = \sum_{n=1}^{p} 2^{n}$.

Proof. Let G be the set of strings representing positions from which the first player can win in the *EXPTIME*-complete game mentioned in FACT 1. It suffices to prove that

$$G \in P \mid poly \Rightarrow G \in \sum_{2}^{p}$$
.

Suppose G = S : h where $S \in P$ and h is polynomial-bounded. Then

$$x \in \mathbf{G} \Leftrightarrow \exists w \forall z [x \in W \cup z \in W \cup (wz \in \mathbf{S} \Leftrightarrow wF_0(z) \\ \notin \mathbf{S} \cup wF_1(z) \notin \mathbf{S})]$$

Here w ranges over all strings of length |h(|x|) and z ranges over all strings of the same length as x. Since membership in S or membership in W can be tested in polynomial time, it tollows that $G \in \sum_{n=1}^{p} b_{n}$.

COROLLARY 6.7. EXPTIME $\subseteq P \neq NP$.

Proof. Assume for contradiction that $EXPTIME \subseteq P/poly$ and P = NP. The first hypothesis implies that $EXPTIME = \sum_{p=2}^{p}$, and the second implies that $P = \sum_{p=2}^{p}$. Hence P = EXPTIME. But this contradicts the result that $P \subseteq EXPTIME$, which is easily proved by diagonalization.

Figure 1. MAIN RESULTS

$$\begin{split} PSPACE &\subseteq P \mid poly \Rightarrow PSPACE = \sum_{2}^{p} \cap \sum_{2}^{p} \\ PSPACE &\subseteq P \mid log \Leftrightarrow PSPACE = P \\ EXPTIME &\subseteq PSPACE \mid poly \Leftrightarrow EXPTIME = PSPACE \\ P &\subseteq DSPACE \left((log n)^{l} \right) \mid log \Leftrightarrow P \subseteq DSPACE \left((log n)^{l} \right) \\ NSPACE \left(log n \right) \subseteq DSPACE \left(log n \right) \mid log \\ \Leftrightarrow NSPACE \left(log n \right) = DSPACE \left(log n \right) \end{split}$$

$$NP \subseteq P / log \Leftrightarrow P = NP \quad (^{1})$$

$$NP \subseteq P / poly \Rightarrow \bigcup \sum_{i}^{p} = \sum_{2}^{p} \quad (^{2})$$

$$EXPTIME \subseteq P / poly \Rightarrow EXPTIME = \sum_{2}^{p} \Rightarrow P \neq NP \quad (^{3})$$

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⁽¹⁾ Obtained jointly with Ravindran Kannan.

^{(&}lt;sup>2</sup>) An improvement by Michael Sipser of an early result of ours.

^{(&}lt;sup>3</sup>) Due to Albert Meyer.