NON-STANDARD MODELS OF PEANO ARITHMETIC

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It is fitting that in a Festschrift honoring Ernst Specker we discuss a construction that combines two branches of logic in which Specker made early and enduring contributions. These are the areas of model theory and algorithmic complexity. It is appropriate in the more immediate sense that the method of model construction presented here had its genesis in two papers of Specker. One of these is the well-known article [1] with McDowell on the existence of end extensions of non-standard models of Peano arithmetic. The other [2] is on the falsity of a recursive form of Ramsey's Theorem. One of the by-products of our construction is a new proof of a strengthened form of the result in [2].

The problem of constructing non-standard models of arithmetic to prove independence results is as old as Gödel's Incompleteness Theorem itself, and has been raised many times in the literature, e.g. in Scott [6]. The possibility immediately arose of proving independence not by the self-reference technique of Gödel but rather by the older model building method used in geometry. In fact the existence of such models is guaranteed by Gödel's Completeness Theorem. In axiomatic set theory these hopes were amply fulfilled through the work of Gödel, Paul Cohen, and others, so that known conjectures in analysis and algebra have by now been shown independent by model building techniques.

In arithmetic the results until recently have been meager and negative in character. For instance S. Tennenbaum proved that non-standard models of Peano arithmetic could not be recursive structures. The best one can attain is a Δ_2^0 ($= \sum_2^0 \cap \prod_2^0$)-model. The recent remarkable results of Paris, Kirby, and Harrington have entirely changed the situation. In Paris and Harrington [3] a modified form of the finite Ramsey Theorem was shown to be independent from Peano arithmetic. The method employed in [3] was to show that this modified Ramsey Theorem implied the consistency of Peano arithmetic and hence was unprovable. Another, earlier

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approach developed by Paris and Kirby ([4] and [5]) used a more model-theoretic method, employing initial segments of models. In the present paper we shall reprove the independence of the modified Ramsey Theorem by the direct mathematical construction of a non-standard model of Peano arithmetic in which this theorem is false. Although this method does lead to a rather direct proof of the Paris and Harrington result, for us the focus of this paper is on the fact that it is now possible to construct non-standard models without any use of metamathematical concepts.

For the last twenty years or so there has existed an algebraic construction of a non-standard model of Peano arithmetic. This is the well-known ultraproduct construction. When applied to the standard model $\mathbf{N}$ the ultrapower $\mathbf{N}^i/D$ yields such a non-standard model. Unfortunately, the ultrapower does the job of being a model of Peano arithmetic a little too well, since $\mathbf{N}^i/D$ is elementarily equivalent to $\mathbf{N}$. Thus the model $\mathbf{N}^i/D$ cannot be used for independence proofs in arithmetic. On the other hand, it does suggest the possibility that a modification of the ultraproduct construction might lead to models which are elementarily inequivalent to $\mathbf{N}$. Now, the ultrafilter D is exactly tailored to reflect the propositional connectives and can scarcely be modified. The other half of the ultrapower, namely the cartesian product $\mathbf{N}^I$, which contains a sufficiently large set of functions $f : I \to \mathbf{N}$ to deal with quantifiers, leaves one with a wider latitude for modification by restricting to a suitable subset $\mathscr{F}$ of $\mathbf{N}^I$. The problem then becomes one of finding a class of functions $f : I \to \mathbf{N}$ for which $\mathscr{F}/D$ is a model of the Peano axioms. We call such a model $\mathscr{F}/D$, where $\mathscr{F} \neq \mathbf{N}^I$, a *restricted ultrapower*. The first non-standard model of Peano arithmetic, constructed by Skolem [7], was in fact a restricted ultrapower, in which the set $\mathscr{F}$ consisted of the first order definable arithmetic functions. However, this model, aside from requiring logical formulas in its definitions, was, like the full ultrapower, elementarily equivalent to $\mathbf{N}$. Other natural candidates for the class $\mathscr{F}$ such as the family of primitive recursive functions or of general recursive functions have been shown to yield restricted ultrapowers which were not models of Peano arithmetic (See Scott [6]). The models we construct are restricted ultrapowers. A surprising aspect of the construction, in view of the aforementioned abortive candidates, is that the family $\mathscr{F}$ of functions we use is closely derived from the class of primitive recursive partitions. However, the functions in $\mathscr{F}$ are defined via sets which are homogeneous for these partitions and these sets are not in general recursive, by Specker's result [2] (or even in $\sum_n^0$, for fixed n, by a theorem of Jockusch [10]).

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Here is a brief description of the simplest model which we construct later. Let $\{P_i\}$ be an effective enumeration of all primitive recursive partitions $P_i : [\mathbf{N}]^{e_i} \to r_i$. It is an immediate consequence of Ramsey's Theorem that there exists a finite set X_k in $\mathbf{N}$, with $\# X_k \geqslant k$, min X_k, which is homogeneous for $P_1, \ldots, P_k$. Let $\{a_{k1}, \ldots, a_{kn_k}\}$ be an enumeration of X_k in increasing order. Let the functions $h_j : \mathbf{N} \to \mathbf{N}$ be defined by

$$h_j(k) = \begin{cases} a_{kj} & j \leqslant n_k \\ h_{j-1}(k)^2 & j > n_k . \end{cases}$$

Now let

$$\mathscr{F} = \{f \mid \exists j \, \forall i \, f(i) \leqslant h_j(i)\} .$$

Then, for any non-principal ultrafilter D, the restricted ultrapower $\mathscr{F}/D$ is a non-standard model of Peano arithmetic. If, in addition, we assume that X_k has been chosen so that a_{kn_k} is a minimum, then the above consequence of Ramsey's Theorem is false in this model.

II. Bounded Ultrapowers

In building the model we have endeavoured to motivate each stage of the construction. Since this is a modification of the ultraproduct construction it is natural to aim at reproducing (to a degree) the main properties of the full ultraproduct. The first property of the ultraproduct we mimic is the Łoš property that a formula is satisfied in the ultraproduct if and only if it is satisfied in a set of factors lying in the ultrafilter. Of course, we wish to have this true for only a limited set of formulas to avoid constructing a model elementarily equivalent to $\mathbf{N}$.

By a *limited formula* we mean one in which every quantifier occurs in bound form: $\forall\, x < z$ or $\exists\, x < z$.

If $f, g \in \mathbf{N}^I$, we write $f \leqslant g$ to mean $f(i) \leqslant g(i)$ for all $i \in I$. A natural constraint on our proposed set $\mathscr{F}$ is that it be closed under $\leqslant$, i.e. $f \leqslant g \in \mathscr{F}$ implies $f \in \mathscr{F}$. We call the restricted ultrapower $\mathscr{F}/D$ resulting from such an $\mathscr{F}$ a *bounded ultrapower*. This condition is sufficient to prove the Łoš property for limited formulas.

The formal language we use for Peano arithmetic has the constant 0 and two binary relation symbols $\sigma(x, y, z)$ and $\pi(x, y, z)$ (denoting $x+y = z$ and $x \cdot y = z$ in $\mathbf{N}$). By not having the functions $+$ and $\cdot$ in the language we avoid having to assume at the outset that $\mathscr{F}/D$ is closed under $+$ and $\cdot$.

THEOREM 1. *Assume* $\mathscr{F}/D$ *is a bounded ultrapower. Let* $\phi(x_1, \ldots, x_n)$ *be a limited formula and* $f_1, \ldots, f_n \in \mathscr{F}$. *Then* $\mathscr{F}/D \models \phi(f_1^*, \ldots, f_n^*)$ *if and only if* $\{ i \mid \mathbf{N} \models \phi(f_1(i), \ldots, f_n(i)\} \in D$.

Proof: We proceed by induction on the length of ϕ. For atomic formulas the equivalence is an immediate consequence of the definition of $\mathscr{F}/D$. That the equivalence is preserved under the logical connectives follows exactly as in the full ultraproduct case from the properties of the ultrafilter D.

Now assume that $\phi(x_1, \ldots, x_n)$ has the form $(\exists x_j < x_k)\,\psi(x_1, \ldots, x_n, x_j)$. Suppose that

$$s = \{i \mid \mathbf{N} \models (\exists x_j < f_k(i))\,\psi(f_1(i), \ldots, f_n(i), x_j)\} \in D\,.$$

Then for each $i \in s$, there exists in $\mathbf{N}$ an element $a_i < f_k(i)$ such that

$$\mathbf{N} \models \psi(f_1(i), \ldots, f_n(i), a_i)\,.$$

Define the functions $g : \mathbf{N} \to \mathbf{N}$ by

$$g(i) = \begin{cases} a_i & \text{for } i \in s \\ 0 & \text{for } i \notin s\,. \end{cases}$$

Since $g < f_k \in \mathscr{F}$, we have $g \in \mathscr{F}$. Now $\{ i \mid \mathbf{N} \models \psi(f_1(i), \ldots, f_n(i), g(i)\} = s \in D$. By the inductive hypothesis

$$\begin{array}{ll} & \mathscr{F}/D \models \psi(f_1^*, \ldots, f_n^*, g^*) \\ \text{or} & \mathscr{F}/D \models (\exists x_j < f_k^*)\,\psi(f_1^*, \ldots, f_n^*, x_j) \\ \text{i.e.} & \mathscr{F}/D \models \phi(f_1^*, \ldots, f_n^*)\,. \end{array}$$

The other half of the equivalence is immediate.

We can extend this result a little further in one direction. The proof of the following consequence is obvious.

COROLLARY. *Let* $\phi(x_1, \ldots, x_n)$ *be a* $\sum_1^0$*-formula. Then*

$$\mathscr{F}/D \models \phi(f_1^*, \ldots, f_n^*) \text{ implies } \{ i \mid \mathbf{N} \models \phi(f_1(i), \ldots, f_n(i)\} \in D\,.$$

The second property of ultraproducts we copy is the saturation property. The ω_1-saturation property of a structure $\mathscr{A}$ is usually formulated as follows. Let $\{ \phi_j(z) \}$ be a countable sequence of formulas with coefficients in $\mathscr{A}$ and the indicated free variable z. Then $\mathscr{A} \models \exists z \bigwedge_{j \leq n} \phi_j(z)$ for every n implies $\mathscr{A} \models \exists z \bigwedge_{j=i}^{\infty} \phi_j(z)$.

An immediate consequence of the saturation property is the following apparently stronger statement: Let $\{ \phi_j (z_1, \ldots, z_{n_j}) \}$ be a countable sequence of formulas with coefficients in $\mathscr{A}$ and with the indicated free variables. Then

$$\mathscr{A} \models \exists z_1 \ldots \exists z_{n_k} \bigwedge_{j \leq k} \phi_j (z_1, \ldots, z_{n_j})$$

for every k implies that

$$\mathscr{A} \models \exists z_1 \ldots \exists z_n \ldots \bigwedge_{j=1}^{\infty} \phi_j (z_1, \ldots, z_{n_j}) .$$

It is this form of the saturation property which we shall adapt in constructing the model $\mathscr{F}/D$. We shall require the property only for a fixed sequence $\{ \phi_j \}$ of limited formulas which we shall specify later in the construction of the model. We shall in addition find it useful to add a condition relating the free variable z_1 with k and n_k in the form of a limited formula $\phi (k, n_k, z_1)$. We could here replace z_1 by a finite sequence of the free variables but we shall not need this added generality.

Let $\phi (x, y, z_1)$ and $\phi_j (z_1, \ldots, z_{n_j})$, $j = 1, 2, 3, \ldots$, be limited formulas with the indicated free variables. We assume that n_j increases with j. Suppose that for each k

$$\mathbf{N} \models \exists z_1 \ldots \exists z_{n_k} (\phi (k, n_k, z_1) \wedge \bigwedge_{j \leq k} \phi_j (z_1, \ldots, z_{n_j}))$$

Given k, let $a_{k1}, \ldots, a_{kn_k} \in \mathbf{N}$ be such that

$$\mathbf{N} \models \phi (k, n_k, a_{k1}) \wedge \bigwedge_{j \leq k} \phi_j (a_{k1}, \ldots, a_{kn_j}) .$$

Define the functions h_j by

$$h_o (k) = n_k \quad \text{for all } k ,$$

and for $j > 0$,

$$h_j (k) = \begin{cases} a_{kj} & \text{for } n_k \geqslant j \\ \text{arbitrary} & \text{for } n_k < j . \end{cases}$$

THEOREM 2. *Let* $\mathscr{F} \subseteq \mathbf{N}^{\mathbf{N}}$ *contain the functions* h_j, $j = 0, 1, 2, \ldots,$ *and* $\mathbf{1}$ *(the identity function) and be closed under* $<$. *Then*

$$\mathscr{F}/D \models \exists x \, \exists y \, \exists z_1 \ldots \exists z_n \ldots (\phi (x, y, z_j) \wedge \bigwedge_{j=1}^{\infty} \phi_j (z_1, \ldots, z_{n_j}))$$

Proof: Since the formula ϕ_m occurs as a conjunct in $\bigwedge_{j \leq n} \phi_j$ for $n_k \geqslant m$, we have $\mathbf{N} \models \phi_m (a_{k1}, \ldots, a_{km})$. i.e. $\mathbf{N} \models \phi_m (h_1 (k), \ldots, h_m (k))$. Thus, by

Theorem 1

$$\mathscr{F}/D \models \phi_m(h_1^*, \dots, h_m^*).$$

Again, for every k

$$\mathbf{N} \models \phi(k, n_k, a_{k1}), \text{ so that}$$

by Theorem 1

$$\mathscr{F}/D \models \phi(\mathbf{1}^*, h_0^*, h_1^*),$$

proving the theorem.

We shall in the sequel be taking for $\mathscr{F}$ the smallest set of functions closed under $<$ and containing $\mathbf{1}$ and all the h_j's. In other words we let

$$\mathscr{F} = \{f \mid f \leqslant \mathbf{1} \quad \text{or} \quad \exists j\, f \leqslant h_j\}.$$

As an example of the use of the Saturation Theorem 2 we show how we can ensure that $\mathscr{F}$ is closed under $+$ and $\cdot$. Since

$$f, g < h_j \quad \text{implies} \quad f + g, f \cdot g < h_j^2$$

it clearly suffices to assume that $h_{j-1}^2 < h_j$. Thus, if we assume that the condition $z_{j-1}^2 < z_j$ occurs in the formula ϕ_j, then, adding $h_j(k) = h_{j-1}^2(k)$ for $j > n_k$ to our definition of h_j, we have that $h_{j-1}^2 < h_j$, so that $\mathscr{F}$ is closed under $+$ and $\cdot$. We shall call this the Closure Condition on $\{h_j\}$.

Again, we can guarantee that $\mathbf{1} \in \mathscr{F}$ by assuming that $\phi(k, n, z_1)$ includes the condition $z_1 > k$.

Up to this point in our construction of $\mathscr{F}/D$ there is no guarantee that the difficulty with the full ultrapower has been obviated. It may happen that $\mathscr{F}/D \equiv N$, so that $\mathscr{F}/D$ cannot be used for independence results. To obtain a true arithmetical statement which is false in $\mathscr{F}/D$ we now add the condition that the sequence $\{\phi_j\}$ of limited formulas is a recursively enumerable set. It follows immediately that the sequence $\{\bigwedge_{j \leq n} \phi_j\}$ is also a recursively enumerable set. Since the satisfaction relation for limited formulas in $\mathbf{N}$ is a primitive recursive relation there is a $\sum_1^0$-formula $\gamma(x, y, z)$ such that

$$\mathbf{N} \models \gamma(k, n_k, z) \leftrightarrow \phi(k, n_k, z_1) \wedge \bigwedge_{j \leq k} \phi_j(z_1, \dots, z_{n_j}).$$

Here $\gamma(k, n_k, z)$ holds if and only if z is a code number of a sequence of length n_k such that $\phi(k, n_k, z_1) \wedge \bigwedge_{j \leq k} \phi_j(z_1, \dots, z_{n_j})$ holds, where $z_i = \beta(i, z)$ and β is the Gödel β-function (as given e.g. in Shoenfield [8] §6.4.)

Suppose that, as in the hypothesis of Theorem 2, we assume that for every k

$$\mathbf{N} \models \exists z_1 \ldots \exists z_{n_k} [\phi(k, n_k, z_1) \wedge \bigwedge_{j \leq k} \phi_j(z_1, \ldots, z_{n_j})]$$

We now construct the functions $\{h_j\}$ of Theorem 2 with greater care. For each k, we choose the sequence $a_{k1}, \ldots, a_{kn_k}$ to be the *least* sequence satisfying

$$\phi(k, n_k, z_1) \wedge \bigwedge_{j \leq k} \phi_j(z_1, \ldots, z_{n_j})$$

in $\mathbf{N}$. The precise measure of what we mean by least is not critical, but we shall take it to mean that the largest element of $a_{k1}, \ldots, a_{kn_k}$ is a minimum for all possible choices of $a_{k1}, \ldots, a_{kn_k}$ satisfying the above formula. We shall henceforth assume by appropriate re-labeling that $a_{k1} < a_{k2} < \ldots < a_{kn_k}$, so that a_{kn_k} is the minimal element. We now claim that

$$\mathscr{F}/D \models \neg \forall k \exists n \exists z_1 \ldots \exists z_{n_k} [\phi(k, n, z_1) \wedge \bigwedge_{j \leq k} \phi_j(z_1, \ldots, z_{n_j})]$$

or, more precisely,

$$\mathscr{F}/D \models \neg \forall k \exists n \exists z \, \gamma(k, n, z).$$

Since $\mathbf{N} \models \forall k \exists z \, \gamma(k, n_k, z)$ we have obtained a true arithmetical statement which is false in $\mathscr{F}/D$. Note that it follows from Theorem 2 that for all $k \in \mathbf{N}$

$$\mathscr{F}/D \models \exists n \exists z \, \gamma(k, n, z).$$

THEOREM 3. $\mathscr{F}/D \models \neg \forall k \exists n \exists z \, \gamma(k, n, z)$.

Proof: Assume that on the contrary

$$\mathscr{F}/D \models \forall k \exists n \exists z \, \gamma(k, n, z).$$

Choose $k = \mathbf{1}^*$. Then there exist $r, g \in \mathscr{F}$ such that

$$\mathscr{F}/D \models \gamma(\mathbf{1}^*, r^*, g^*).$$

By the Corollary to Theorem 1 we have for an infinite set t of k's (lying in the ultrafilter D)

$$\mathbf{N} \models \gamma(k, r(k), g(k)) \tag{1}$$

Now since $\beta(r, g) \leqslant r$ [1] $\in \mathscr{F}$, there exists an i such that

$$\beta(r, g) < h_i.$$

[1]) See Shoenfield [8] § 6.4.

Choose $k > i$ lying in t. Then $h_i < h_k$, so that

$$g_{r(k)}(k) = \beta(r(k), g(k)) < h_k(k) = a_{kn_k}.$$

But by (1) the sequence

$$g_1(k), \ldots, g_{r(k)}(k)$$

satisfies the above formula. This contradicts the minimality of a_{kn_k}.

III. Peano arithmetic and the Stability Condition

Theorem 1 suffices to construct a non-standard model of a theory of arithmetic in which all the axioms are expressed by limited formulas. The induction axioms of Peano arithmetic however involve arbitrary elementary formulas. To deal with this problem we shall associate with each formula $\phi(y)$ of arithmetic a limited formula $\hat{\phi}(y; z)$ [1] called the *limited associate* of $\phi(y)$.

We assume that $\phi(y)$ has been reduced to prenex normal form. To obtain the formula $\hat{\phi}(y; z)$ we replace each quantifier Qx_i in $\phi(y)$ by the bounded quantifier $Qx_i < z_i$. The bounding variables z_k are to be distinct from the variables occurring in $\phi(y)$ and also distinct from each other.

Although Theorem 1 allows us to prove the validity of limited associates of the Peano axioms in the model $\mathscr{F}/D$, we need a provision for inferring from this the validity of the Peano axioms themselves in $\mathscr{F}/D$.

To obtain the desired result it would suffice to show that for some suitable vector h in $\mathscr{F}$, $\mathscr{F}/D \models \hat{\phi}(y; h^*)$ implies $\mathscr{F}/D \models \phi(y)$. However, if we consider the case where $\phi(y)$ is $(\forall x)(y \neq x)$, we find $\mathscr{F}/D \models \hat{\phi}(h^*, h^*)$ but $\mathscr{F}/D \models \neg\phi(h^*)$. This example shows that we must restrict the range of y, i.e. require that for all $f < h$, $\mathscr{F}/D \models \hat{\phi}(f^*, h^*)$ implies $\mathscr{F}/D \models \phi(f^*)$. To prove $\mathscr{F}/D \models \phi(f^*)$ for all f in $\mathscr{F}$ it thus suffices to construct an increasing sequence $\{h_i\}$ in $\mathscr{F}$, cofinal in $\mathscr{F}$, such that for all i, j with $i < j$ [2] and all $f \in \mathscr{F}$ with $f < h_i$,

[1]) Here and later y and z denote vectors of variables.

[2]) Here and later j denotes a vector $\langle j_1, \ldots, j_n \rangle$, $i < j$ means $i < \min j$, and h_j denotes the vector $\langle h_{j_1}, \ldots, h_{j_n} \rangle$.

$$\mathscr{F}/D \models \hat{\phi}(f^*; h_j^*) \text{ if and only if } \mathscr{F}/D \models \phi(f^*). \tag{1}$$

Now the equivalence (1) refers to a formula $\phi(y)$ with unbounded quantifiers and so is not a tractable condition to handle via Theorems 1 and 2. We shall accordingly replace (1) by an equivalent condition which refers only to limited formulas. To see what this condition is consider the case where $\phi(y)$ is a formula of the form $(\forall x)\,\psi(x, y)$, where ψ is quantifier free. Suppose that for some i, j with $i < j$ and all $f \in \mathscr{F}$ with $f < h_i$ we have

$$\mathscr{F}/D \models \hat{\phi}(f^*; h_j^*)$$

$$\text{i.e.} \quad \mathscr{F}/D \models (\forall x < h_j^*)\,\psi(f^*, x)$$

If condition (1) holds then

$$\mathscr{F}/D \models (\forall x)\,\psi(f^*, x)$$

and hence

$$\mathscr{F}/D \models (\forall x < h_{j'}^*)\,\psi(f^*, x)$$

for all $j' > i$.

Thus, for condition (1) to hold it is necessary for the truth value of the limited formula $\hat{\phi}(y; z)$ to eventually stabilize. We formulate this condition as follows.

We shall assume for the rest of the paper that the sequence $h_j = \langle h_{j1}, \ldots, h_{jn} \rangle$ substituted for the bounding variables $z = \langle z_{i_1}, \ldots, z_{i_n} \rangle$ in the limited formula $\psi(y; z)$ is an increasing sequence (i.e. $r < s$ implies $j_r < j_s$ and hence $h_{j_r} < h_{j_s}$). Thus, the smaller the scope of the bounded quantifier $(Qx_{i_r} < z_{i_r})$ in $\psi(y; z)$ the larger the substituted element $h_{j_r}^*$ in the sequence $h_j^* = \langle h_{j1}^*, \ldots, h_{jn}^* \rangle$.

Stability Condition. For every limited formula $\psi(y; z)$ and for all i, j, j' with $i < j$, $i < j'$

$$\mathscr{F}/D \models (\forall y < h_i)\,(\psi(y; h_j^*) \leftrightarrow \psi(y; h_{j'}^*)) .$$

We shall now prove that this condition suffices to establish (1) (c.f. [3] Proposition 2.2).

LEMMA 1. *Assume that $\{h_i\}$ is an increasing cofinal sequence in $\mathscr{F}$ which satisfies the Stability Condition. Then for all $i < j$*

$$\mathscr{F}/D \models (\forall y < h_i^*)\,(\phi(y; h_j^*) \Leftrightarrow \hat{\phi}(y; h_j^*)) .$$

Proof: We shall assume that $\phi(y)$ is already in prenex normal form. We proceed by induction on the number of quantifiers occurring in $\phi(y)$. For quantifier-free formulas the equivalence is clearly true.

Now assume that $\phi(y)$ has the form $(\exists x_1)\,\psi(x_1, y)$. Then $\hat{\phi}(y; z)$ has the form $(\exists x_1 < z_1)\,\hat{\psi}(x_1, y; z_2, \ldots, z_e)$. $\mathscr{F}/D \models \phi(f^*)$ if and only if $\mathscr{F}/D \models \hat{\psi}(b^*, f^*)$ for some $b \in \mathscr{F}$.

By induction it follows that $\mathscr{F}/D \models \phi(f^*)$ if and only if $\mathscr{F}/D \models \hat{\psi}(b^*, f^*, h^*_{j_2}, \ldots, h^*_{j_e})$ for all $j_2, \ldots, j_e$, with $i < j_2 < \ldots < j_e$. By cofinality $b < h_{j_1}$, for some $j_1 > i$. Hence

$$\mathscr{F}/D \models \phi(f^*) \text{ if and only if } \mathscr{F}/D \models (\exists x_1 < h^*_{j_1})\,\hat{\psi}(x_1, f^*; h^*_{j_2} \ldots h^*_{j_e})$$
$$\text{i.e. } \mathscr{F}/D \models \hat{\phi}(f^*; h^*_{j_0}), \text{ where } j_o = \langle j_1, j_2, \ldots, j_e \rangle .$$

By the Stability Condition, $\mathscr{F}/D \models \phi(f^*; h^*_{j_0})$ for this $j_o > i$ is equivalent to $\mathscr{F}/D \models \hat{\phi}(f^*; h^*_j)$ for all $j > i$, completing the induction step.

THEOREM 4. *Assume that* $\{h_i\}$ *is a cofinal sequence in* $\mathscr{F}$ *with* $h_i^2 < h_{i+1}$ *which obeys the Stability Condition. Then* $\mathscr{F}/D$ *is a model of the Peano axioms.*

Proof. The axioms $\forall x\, \forall y\, \exists z\, \sigma(x, y, z)$ and $\forall x\, \forall y\, \exists z\, \pi(x, y, z)$ are valid because $\mathscr{F}$ is closed under $+$ and $\cdot$.

Every other non-induction Peano axiom ϕ is a $\prod^0_1$-statement. Thus $\mathbf{N} \models \hat{\phi}(z)$. By Theorem 1, $\mathscr{F}/D \models \hat{\phi}(z)$. Hence $\mathscr{F}/D \models \phi$.

Now let $\phi(y)$ be the induction formula

$$[\psi(0, y) \wedge (\forall x)(\psi(x, y) \to \psi(x+1, y))] \to (\forall x)\,\psi(x, y) \ .$$

We may assume that $\psi(x, y)$ is in prenex normal form.

Note that for any formula $\eta(x)$

$$\mathbf{N} \models [\eta(0) \wedge (\forall x < w)(\eta(x) \to \eta(x+1))] \to (\forall x < w)\,\eta(x) \,.$$

Hence, if $\eta(x)$ is a limited formula then Theorem 1 implies that

$$\mathscr{F}/D \models [\eta(0) \wedge (x < w)(\eta(x) \to \eta(x+1))] \to (\forall x < w)\,\eta(x) \ .$$

In particular, taking for η the limited associate $\hat{\psi}(x, y; z)$ of $\psi(x, y)$, we have that

$$F/D \models [\hat{\psi}(0, y; z) \wedge (\forall x < w)(\hat{\psi}(x, y; z) \to \hat{\psi}(x+1, y; z)]$$
$$\to (\forall x < w)\,\hat{\psi}(x, y; z) \qquad (1)$$

We now assume that $\mathscr{F}/D \models \psi(0, g^*)$ and $\mathscr{F}/D \models (\forall x)(\psi(x, g^*)) \to \psi(x+1, g^*))$ for some vector g of functions in $\mathscr{F}$.

We have $g < h_i$ (i.e. $\max g < h_i$) for some i. Choose any j, t with $j > t > i$. By Lemma 1, $\mathscr{F}/D \models \psi(0, g^*; h_j^*)$. Assume for $x < h_t^*$ that

$$\mathscr{F}/D \models \hat{\psi}(x, g^*; h_j^*)\,.$$

By Lemma 1,

$$\mathscr{F}/D \models \psi(x, g^*)\,.$$

Hence,

$$\mathscr{F}/D \models \psi(x+1, g^*)$$

so that, again by Lemma 1,

$$\mathscr{F}/D \models \hat{\psi}(x+1, g^*; h_j^*)\,.$$

Thus, by (1),

$$\mathscr{F}/D \models (\forall x < h_t^*)\,\hat{\psi}(x, y^*; h_j^*)\,.$$

It follows from Lemma 1 that

$$\mathscr{F}/D \models \forall x\, \psi(x, g^*)\,.$$

We have thus proved that the induction formula $\phi(y)$ is valid in $\mathscr{F}/D$.

It remains for us to construct a suitable sequence $\{h_i\}$ of functions satisfying the Stability Condition. Let $\{\psi_j\}$ be an effective enumeration of all the limited formulas. The Stability and Closure Conditions have the form

$$\mathscr{F}/D \models \exists z_1 \ldots \exists z_n \ldots \bigwedge_{\substack{1 \leqq i < j,\, j' < \infty \\ 1 \leqq s < \infty}} [(\forall z < z_i)(\psi_s(y; z_j)$$
$$\leftrightarrow \psi_s(y; z_{j'})) \wedge z_{j-1}^2 < z_j]$$

Now this condition has precisely the form needed for the conclusion of the Saturation Theorem 2. Thus, if we can show that for each k

$$\mathbf{N} \models \exists z_1 \ldots \exists z_{n_k} \bigwedge_{\substack{1 \leqq i < j,\, j' < n_k \\ 1 \leqq s < k}} [(\forall y < z_i)(\psi_s(y; z_j)$$
$$\leftrightarrow \psi_s(y; z_{j'})) \wedge z_{j-1}^2 < z_j] \qquad (*)$$

then we can construct the sequence $\{h_i\}$ and the set $\mathscr{F}$ to satisfy the Stability and Closure Conditions via Theorem 2.

We could now proceed to show that the above condition is indeed satisfied in $\mathbf{N}$ and thus construct a non-standard model of Peano arithmetic. However, our goal is the construction of a mathematically perspicuous model which is independent of the logical formulas. The functions $\{h_i\}$ given by the above condition require the logical calculus in their definition. Accordingly, we shall consider a larger class $\mathscr{F}$ of functions than those defined above, which we shall construct independently of logical formulas. This class will be constructed from combinatorial principles derived from Ramsey's Partition Theorem.

IV. Ramsey-type Theorems

The infinite Ramsey Theorem states that for every partition $P : [\mathbf{N}]^e \to r$ [1]) there exists an infinite subset X of $\mathbf{N}$ such that $P \mid [X]^e$ is constant. In these circumstances one says that X is homogeneous for the partition P. This set-theoretic theorem has various combinatorial consequences which are formalizable in elementary arithmetic. One such immediate consequence which we shall prove independent of the Peano axioms is the following.

Proposition 1. *Let $P : [\mathbf{N}]^e \to r$ be a primitive recursive partition. For every natural number k there exists a finite subset X of $\mathbf{N}$, with $\# X \geqslant k$ and $\# X \geqslant 2^{2^{\min X}}$, which is homogeneous for the partition P.*

In order to apply Theorem 2 we require the construction of a set which is simultaneously homogeneous for several partitions. This is easily done by the infinite Ramsey Theorem. Suppose $P_1 : [\mathbf{N}]^{e_1} \to r_1$ and $P_2 : [\mathbf{N}]^{e_2} \to r_2$ are two partitions. Let X_1 be an infinite subset of $\mathbf{N}$ homogeneous for P_1. Then $P_2 \mid [X]^{e_2}$ is a partition of $[X_1]^{e_2}$, and hence there is is an infinite subset X_2 of X_1 which is homogeneous for P_2 (as well as P_1). This proof extends immediately to finitely many partitions. A direct consequence is the following generalization of Proposition 1.

Proposition 2. *Let $P_i : [\mathbf{N}]^{e_i} \to r_i$, $i \leqslant i \leqslant n$ be a set of primitive recursive partitions. For every natural number k there exists a finite subset X of $\mathbf{N}$ with $\# X \geqslant k$ and $\# X \geqslant 2^{2^{\min X}}$, which is simultaneously homogeneous for all the partitions $P_1, \ldots, P_n$.*

---

[1]) We identify the number r with the set of all natural numbers $< r$.

Proposition 2 may be expressed by a $\prod_2^0$ formula. First it is clear that we can construct a $\sum_1^0$-formula ϕ_i that expresses the properties that

1. $P_i : [\mathrm{N}]^{e\,i} \to r_i$ is a primitive recursive partition
2. $z_1 < z_2 < \dots < z_{n_k}$
3. $\{ z_1, \dots, z_{n_k} \}$ is homogeneous for P_i
4. $k \leqslant n_k$
5. $2^{2^{z_1}} \leqslant n_k$

Proposition 2 asserts that for every k

$$\mathbf{N} \models \exists z_1 \dots \exists z_{n_k} \bigwedge_{i \leq k} \phi_i \,.$$

V. Construction of the Model

We now have all the ingredients at hand to construct a non-standard model of Peano arithmetic, and we have only to assemble them according to the specifications of Section II.

Let P_i be an effective enumeration of all primitive recursive partitions $P_i : [\mathrm{N}]^{e\,i} \to r_i$. By Proposition 2 we have that for every k

$$\mathbf{N} \models \exists z_1 \dots \exists z_{n_k} \bigwedge_{i \leq k} \phi_i$$

where ϕ_i is the $\sum_1^0$-formula of Section IV expressing the conditions (1)-(5) satisfied by the partition P_i.

Following the prescription given in Section III we let a_{kn_k} be the smallest number such that $a_{k1}, \dots, a_{kn_k}$ is an increasing sequence satisfying the formula $\bigwedge_{j \leq k} \phi_j$. Now we define the functions h_j by

$$h_0(k) = n_k \quad \text{for every } k$$

and for $j > 0$

$$h_j(k) = \begin{cases} a_{kj} & \text{for } j \leqslant n_k \\ h_{j-1}(k)^2 & \text{for } j > n_k \,. \end{cases}$$

Let $\mathscr{F} = \{ f \mid f \leqslant h_j \}$.

Since $\mathbf{1} \leqslant h_o$ the function $\mathbf{1}$ is automatically in $\mathscr{F}$.

By Theorem 2 the sequence $\{ h_j \}$ satisfies $\bigwedge_{j<\infty} \phi_j$ in $\mathscr{F}/D$. We now prove that this implies that the sequence $\{ h_j \}$ satisfies the Stability and

Closure Conditions in $\mathscr{F}/D$. As we saw in Section III it suffices for this purpose to show that for each k

$$\mathbf{N} \vdash \exists z_1 \ldots \exists z_{n_k} \bigwedge_{\substack{1 \leq i < j, j' < n_k \\ 1 \leq s < k}} [(\forall y < z_i)(\psi_s(y; z_j) \leftrightarrow \psi_s(y; z_{j'}) \wedge z_{j-1}^2 < z_j] \tag{*}$$

Let t_i be the length of the sequence y in $\psi_i(y; z)$. Define the partitions $T : [\mathrm{N}]^2 \to 2$, $Q_i : \mathrm{N} \to t_i^2 + 1$, and $S_i : [\mathrm{N}]^{2e+1} \to 2$ by :

$$T(a, b) = \begin{cases} 1 & \text{if } a^2 < b \\ 0 & \text{if not} \end{cases}$$

$$Q_i(a) = \min(a, [t_i \log_2 a] + 1)$$

and for $a \in \mathrm{N}$, $c, c' \in [\mathrm{N}]^e$

$$S_i(a, c, c') = \begin{cases} 1 & \text{if } (\forall y < a)(\psi_i(y; c) \leftrightarrow \psi_i(y; c')) \\ 0 & \text{if not.} \end{cases}$$

The partitions T, Q_i, and S_i are clearly primitive recursive since $\psi_i(y; z)$ is a limited formula. Hence T, $Q_1, \ldots, Q_k$, $S_1, \ldots, S_k$ occur in the sequence $\{P_i\}$. Thus by looking sufficiently far in the sequence we can find a set $X_k = \{a_{k1}, \ldots, a_{kn_k}\}$ which is homogeneous for T, $Q_1, \ldots, Q_k$, $S_1, \ldots, S_k$ with $n_k \geqslant k, 2^{2^{a_{k1}}}$.

Since $a_{kn_k} > a_{k1}^2$, $T(a_{k1}, a_{kn_k}) = 1$. Hence, by homogeneity,

$$T(a, b) = 1, \quad \text{i.e.} \quad a^2 < b,$$

for all $a < b$ in X_k.

Since $\# X > 1$, and X_k is homogeneous for Q_i, $a_{k1} \geqslant t_i \log_2 a_{k1}$.

The number of sequences of numbers $< a_{k1}$ of length t_i is $< a_{k1}^{t_i}$. The number of distinct sequences of truth values of length $a_{k1}^{t_i}$ is $< 2^{a_{k1}^{t_i}}$. Now $n_k > 2^{2^{a_{k1}}} > 2^{a_{k1}^{t_i}}$ since $a_{k1} \geqslant t_i \log_2 a_{k1}$. Thus there are distinct $c, c' > a$ in. X_k such that

$$(\forall y < a_{k1}) \ (\psi_i(y; c) \leftrightarrow \psi_i(y; c')),$$

i.e. $S_i(a_{k1}, c, c') = 1$.

By homogeneity

$$S_i(a, b, b') = 1 \qquad \text{for } all \qquad a < b, b' \text{ in } X_k$$

proving (*).

We have thereby shown that $\mathscr{F}/D$ is a model of the Peano axioms. Since a_{kn_k} was chosen minimal, Proposition 2 is false in $\mathscr{F}/D$, and hence independent of the Peano axioms.

Proposition 1 is also false in $\mathscr{F}/D$. In fact it is provable in Peano arithmetic that Proposition 1 implies Proposition 2. This is a consequence of the following lemma, provable in Peano arithmetic (c.f. Lemma 2.9 in [3]).

LEMMA 2. *Let* $P_i : [\mathrm{N}]^{e_i} \to r_i$, $1 \leqslant i \leqslant n$, *be* n *partitions. There is a partition* $P : [\mathrm{N}]^e \to r$ *such that for all subsets* H *of* N *of cardinality* $> e$, H *is homogeneous for* P *if and only if* H *is homogeneous for all the* P_i.

We may also obtain a purely finitary combinatorial principle which is false in our model.

PROPOSITION 3. *For all natural numbers* e, r, *and* k *there exists an* N, *such that for all partitions* $P : [N]^e \to r$ *there exists a subset* X *of* N, *with* $\# X \geqslant k$ *and* $\# X \geqslant 2^{2^{\min X}}$, *which is homogeneous for* P.

This result follows immediately from the infinite Ramsey Theorem by an application of König's Lemma. If we drop the condition that $\# X \geqslant 2^{2^{\min X}}$, then we obtain the usual finite Ramsey Theorem. Ramsey [11] gave a proof of the latter theorem which is formalizable in Peano arithmetic. Proposition 3 directly yields Proposition 1, for if $P : [\mathrm{N}]^e \to r$ is a partition and k is a number then by considering the partition $P \mid [N]^e$, where N is the number provided by Proposition 3 we obtain the required homogeneous set X for $P \mid [N]^e$ and hence for P. This proof may be carried out in Peano arithmetic. Thus, Proposition 3 is false in our model and independent of the Peano axioms.

VI. A SIMPLER MODEL

The condition in Proposition 1 that $\# X \geqslant 2^{2^{\min X}}$ can be simplified and so yield a simpler sequence $\{h_i\}$ of functions which define the model $\mathscr{F}/D$. In this section we describe such a model by using a combinatorial consequence of Ramsey's Theorem wich is closer to the proposition proved independent in [3].

PROPOSITION 4. *Let* $P : [\mathrm{N}]^e \to r$ *be a primitive recursive partition. For every* k *there exists a finite subset* X *of* N, *with* $\# X \geqslant k$ *and* $\# X \geqslant \min X$, *which is homogeneous for the partition* P.

Proposition 4 implies Proposition 1 via the following result, the proof of which is the same as the proof of Lemma 2.14 of [3].

LEMMA 3. *Let* $P : [\mathbf{N}]^e \to r\ (e \geqslant 2)$ *be a partition. There is a partition* $P^* : [\mathbf{N}]^e \to r^*$ *(where* r^* *depends only on* m, e, *and* r*) such that if* X^* *is a finite subset of* $\mathbf{N}$, *homogeneous for* P^* *with* $\# X^* \geqslant e+1$ *and* $\# X \geqslant \min X$, *then the set* $X = [\log_2 \log_2](X^*)$ *is homogeneous for* P, *and*

$$\# X \geqslant e+1 \quad \text{and} \quad \# X \geqslant 2^{2^{\min X}} .$$

Moreover, if P *is a primitive recursive partition, then* P^* *can be chosen to be primitive recursive.*[1]

Since this proof that Proposition 4 implies Proposition 1 may be carried out in Peano arithmetic, it follows that Proposition 4 is also false in our model $\mathcal{F}/D$. However, our aim here is not merely to give a simple independent statement but to construct a simpler model for Peano arithmetic. Once again we actually use a version of the combinatorial principle which applies to several partitions. The following result is implied by Proposition 4 in Peano arithmetic.

PROPOSIIION 5. *Let* $P_i : [\mathbf{N}]^{e_i} \to r_i,\ 1 \leqslant i \leqslant n$, *be a set of primitive recursive partitions. For every* k *there exists a finite subset of* $\mathbf{N}$, *with* $\# X \geqslant k$ *and* $\# X \geqslant \min X$, *which is simultaneously homogeneous for all the partitions* $P_1, \ldots, P_n$.

We now construct a non-standard model via Proposition 5. Let $\{P_i\}$ again be an effective enumeration of all the primitive recursive partitions $P_i : [\mathbf{N}]^{e_i} \to r_i$. Let $c_{k1}, \ldots, c_{kn_k}$ be an increasing sequence with c_{kn_k} the least number such that $c_{k1}, \ldots, c_{kn_k}$ is homogeneous for all $P_1, \ldots, P_k$, with $c_{k1} \leqslant n_k$ and $k \leqslant n_k$. Define the functions g_j by

$$g_0(k) = n_k \quad \text{for every } k$$

and for $j > 0$

$$g_j(k) = \begin{cases} c_{kj} & \text{for } j \leqslant n_k \\ g_{j-1}(k)^2 & \text{for } j > n_k . \end{cases}$$

Let $\mathcal{F} = \{f \mid \exists j f \leqslant g_j\}$.

We shall show that $\mathcal{F}/D$ is a model of Peano arithmetic by proving that there is an increasing sequence $\{h_j\}$ which lies in and is cofinal with $\mathcal{F}$ and which satisfies the Stability and Closure Conditions. We set

$$h_j = [\log_2 \log_2 g_j] .$$

[1]) Here, as is customary, $[x]$ is the greatest integer $\leqslant x$.

Since $h_j < g_j$, $h_j \in \mathscr{F}$. It follows from Lemma 2.13 of [3] that there is a primitive recursive partition R such that if X is homogeneous for R, with $\# X \geqslant \min X$ and $\# X \geqslant 3$, then for every $x, y, \in X$, $x < y$ implies $2^{2^x} < y$. Since this partition appears in the enumeration $\{ P_i \}$ at some point k, it follows that, for all $i \geqslant k$ and $j < n_i$, $2^{2^{g_j(i)}} < g_{j+1}(i)$. Thus, if for a given j we choose an $m \geqslant k$ such that $n_m \geqslant j$, then, for all $i \geqslant m$, $2^{2^{g_j(i)}} < g_{j+1}(i)$. For every $i < m$ choose an s_i with $2^{2^{g_j(i)}} < g_{s_i}(i)$. Let

$$s = \max (s_1, \ldots, s_{m-1}, j+1)$$

Then

$$2^{2^{g_i}} < g_s \ .$$

Thus $h_s = [\log_2 \log_2 g_s] > g_j$, proving that $\{ h_i \}$ is cofinal in $\mathscr{F}$.

For each partition P_k in the sequence $\{ P_i \}$ there exists another partition $P_t \, (= P_k^*)$ satisfying the conditions of Lemma 3. By the definition of the functions g_j, the set $\{ g_1(t), \ldots, g_{n_t}(t) \}$ is homogeneous for P_t and $n_t \geqslant t$, $n_t \geqslant g_1(t)$. Hence, by Lemma 3, the set

$$\{ h_1(t), \ldots, h_{n_t}(t) \} = \{ [\log_2 \log_2 g_1(t)], \ldots, [\log_2 \log_2 g_{n_t}(t)] \}$$

is homogeneous for P_k and $n_t \geqslant 2^{2^{h_1(t)}}$. Thus, as in the previous section, the sequence $\{ h_j \}$ fulfills the conditions which ensure the satisfaction of the Stability and Closure Conditions. This proves that $\mathscr{F}/D$ is a model of the Peano axioms. Once again, since c_{kn_k} was chosen as minimal, it follows that Proposition 5, and hence Proposition 4, is false in $\mathscr{F}/D$, and therefore independent of Peano arithmetic.

As before we may formulate a finite version of this combinatorial principle.

PROPOSITION 6. *For every* e, r, *and* k *there exists an* N *such that for every partition* $P : [N]^e \to r$ *there exists a subset* X *of* N, *with* $\# X \geqslant k$ *and* $\# X \geqslant \min X$, *which is homogeneous for* P.

Again it is provable in Peano arithmetic that Proposition 6 implies Proposition 4, so that Proposition 6 is false in our model. Proposition 6 was first proved independent of Peano arithmetic in [3] by showing that it implies the consistency of Peano arithmetic and then applying Gödel's Theorem.

Let $C_k = \{ i \mid i \leqslant c_{kn_k} \}$. The model $\mathscr{F}/D$ is an initial segment not only of the ultrapower $\mathbf{N}^I/D$ but also of the smaller ultraproduct $\prod_{k \in \mathbf{N}} C_k/D$.

This indicates that the function C given by $C(k) = c_{kn_k}$ is a very rapidly growing function. In fact the function C majorizes every recursive function which is a provably total function in Peano arithmetic.

THEOREM 5. *Let f be a recursive function. Let ψ be an elementary statement expressing the condition that f is a total function. If ψ is provable in Peano arithmetic, $f(k) < C(k)$ for all sufficiently large k.*

Proof. Suppose $t = \{k \mid f(k) \geqslant C(k)\}$ is infinite. Let D be a non-principal ultrafilter such that $t \in D$. Then $f^* \geqslant C^*$. On the other hand, $f^* = f(\mathbf{1}^*) \in \mathscr{F}/D$, so that $f^* < C^*$, a contradiction.

It follows a fortiori that if N is the smallest integer to satisfy Theorem 5 then this function N also majorizes every provably total recursive function (c.f. Theorem 3.2 in [3]).

We mentioned in the introduction that a by-product of our construction is a new proof of Specker's theorem that there exists a recursive partition with no recursively enumerable infinite homogeneous set. In fact we may obtain the stronger theorem that for each $e \geqslant 2$, there exists a primitive recursive partition: $P : [\mathrm{N}]^e \to 2$ such that P has no infinite homogeneous set in $\sum_e^0$ (c.f. Jockusch [10], Theorem 5.1). We outline the proof of this result. Let $\phi(y)$ be any formula. As in Section III, the limited associate $\hat{\phi}(y; z)$ of $\phi(y)$ defines a partition $P : [\mathrm{N}]^e \to 2$ such that every sequence $\{b_i\}$ of natural numbers homogeneous for P satisfies the Stability Condition for $\hat{\phi}(y; z)$ in N. Hence, for any vector a in N $\phi(a)$ holds in N if and only if $\hat{\phi}(a; b)$ does. It follows that the set $\{a \mid \mathrm{N} \models \phi(a)\}$ is recursive in the set $\{b_i\}$. Thus the set $\{b_i\}$ is not in $\sum_e^0$.

VII. VARIATIONS

We conclude with a series of remarks on various modifications of our construction.

(a) It is easily proved that if $\mathscr{F}$ is closed under $<$ and contains $\mathbf{1}$, then $\mathscr{F}/D$ is non-denumerable, for every non-principal ultrafilter D. Thus, this construction leads only to non-denumerable models. However, a slight variation of the basic construction yields denumerable models. Note that in the proof of Theorem 1 the function g is primitive recursive in f. It follows that we may define $\mathscr{F} = \{f \mid \exists j f \leqslant h_j$ and f is primitive recursive in $h_j\}$. The equivalence $f \equiv g$ in $\mathscr{F}$ defined by a non-principal ultrafilter D, viz $\{ i \mid f(i) = g(i) \} \in D$, may now be directly defined by a $\prod_2^0$-formula. This shows how to construct $\prod_2^0$-models of Peano arithmetic in the form of restricted ultrapowers.

(b) We may reduce the size of our models even further. Since in the proof of Theorem 1 the function g is defined from f by means of a limited formula g is even elementary recursive in f in the sense of Kalmar (see Kleene [11] §57 for a definition of elementary recursive functions). Thus, we may take $\mathscr{F}$ to consist of all functions which are elementary recursive in and majorized by a function h_j, for some j. Moreover, since the functions P_i from which the functions h_j are derived (Section IV) are defined by means of limited formulas, we may also take our sequence of partitions $\{ P_i \}$ to consist of elementary recursive partitions rather than primitive recursive partitions.

(c) It is possible to give the ultrapower a more algebraic appearance by switching from models of $\mathbf{N}$ to models of the ring $\mathbf{Z}$ of all integers. Let T be an axiom system for an ordered ring in which the non-negative elements obey the Peano axioms. Define the functions h_j as in Section V (or the g_j of Section VI). Let $\mathscr{F}$ be the ring of all functions $f: \mathbf{N} \to \mathbf{Z}$ such that, for some j, $|f| \leqslant h_j$. It is easily seen, as in Scott [6], that the ultrafilters are in one-one correspondence $D \leftrightarrow P_D$, with the minimal prime ideals P_D in $\mathscr{F}$, such that $\mathscr{F}/D = \mathscr{F}/P_D$. Principal ultrafilters correspond to principal prime ideals. Thus, we may construct non-standard models of $\mathbf{Z}$ by dividing the ring $\mathscr{F}$ by a non-principal minimal prime ideal P in $\mathscr{F}$. A non-standard model of $\mathbf{N}$ may then be selected as the set of those elements in $\mathscr{F}/P$ which are representable as the sum of four squares.

(d) It is possible to by-pass the Stability Condition in defining a non-standard model $\mathscr{F}/D$. It was condition (1) of Section III that assured us that $\mathscr{F}/D$ was a model of the axioms. We may define the family $\mathscr{F}$ to guarantee that condition (1) holds in a direct manner. We outline this approach now. Reduce the conjunction of the first k axioms to prenex normal form ϕ_k. We may associate with ϕ_k a sequence $f_{k1}, \ldots, f_{kn_k}$ of Skolem functions in the usual manner. For each k define the sequence of natural numbers $a_{k1}, \ldots, a_{kk}$ by induction. Let

$$a_{k1} = k + 1$$

and, for $1 < j < k$, let

$a_{k\,j+1}$ = any number greater than a_{kj}^2 and the values of $f_{k1}, \ldots, f_{kn_k}$ as the arguments range over values $< a_{kj}$.

We now define the functions h_j as before by

$$h_j(k) = \begin{cases} a_{kj} & \text{for } j \leqslant k \\ h_{j-1}^2(k) & \text{for } j > k\,. \end{cases}$$

A simple induction argument now shows that (1) holds for the axioms and hence that $\mathscr{F}/D$ is a model of Peano arithmetic. If as in Theorem 3 the a_{kk} is chosen as the least number so that the sequence $a_{k1}, \ldots, a_{kk}$ satisfies the above conditions then a true statement which is false in $\mathscr{F}/D$ may be constructed via the method of Theorem 3.

The disadvantage of this direct approach is that the model $\mathscr{F}/D$ constructed in this manner is dependent in its definition upon logical formulas and so is not as purely an algebraic construction. Moreover the independent statement which results has no simple combinatorial expression as have those given in Sections IV, V, and VI. Note that in this approach we have not used the property peculiar to Peano's axioms concerning the limited associates of the axioms which is expressed in the proof of Theorem 4. This shows that the method outlined here applies to any recursively enumerable set of axioms for arithmetic which is sufficient to allow the coding required for Theorem 3. Thus, we may prove a general form of Gödel's Incompleteness Theorem without the use of self-reference techniques. At the same time the very generality of the approach outlined here indicates that there is no hope by this method to avoid the use of metamathematics. It is only the above-mentioned property of the Peano axioms vis-à-vis limited formulas that allowed us the latitude to define suitable functions h_j, and hence the model $\mathscr{F}/D$, by means of a combinatorial principal without reference to logical formulas.

Note (Added in proof)

The first sentence of the section entitled "Added in proof" of Kochen and Kripke [12] p. 294, which was inserted by the second author, is not correct and should be deleted in favor of the following corrected version. The first author proposed that the Paris-Harrington statement is false in an initial segment of any non-standard model, and this was verified jointly by the two authors. Adapting this idea, the second author defined the set $\mathscr{F}$ of functions which result in the model of Section V. The first author subsequently found the new set $\mathscr{F}$ of functions which define the simpler model of Section VI.

The devices used in Section III are an adaptation of the ideas in Paris-Harrington [3].

The approach outlined in (d) of Section VII is due to the second author and leads to a concept of 'satisfying' formulas by finite sequences called *fulfillability* wich leads to model-theoretic proofs of many theorems (such as Gödel's and Rosser's theorems) usually proved proof-theoretically and to other applications to the model theory and proof theory of arithmetic. It will be developed in a subsequent paper of the second author.

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