IV. Ramsey-type Theorems

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then we can construct the sequence $\{h_i\}$ and the set $\mathscr{F}$ to satisfy the Stability and Closure Conditions via Theorem 2.

We could now proceed to show that the above condition is indeed satisfied in **N** and thus construct a non-standard model of Peano arithmetic. However, our goal is the construction of a mathematically perspicuous model which is independent of the logical formulas. The functions $\{h_i\}$ given by the above condition require the logical calculus in their definition. Accordingly, we shall consider a larger class $\mathscr{F}$ of functions than those defined above, which we shall construct independently of logical formulas. This class will be constructed from combinatorial principles derived from Ramsey's Partition Theorem.

IV. Ramsey-type Theorems

The infinite Ramsey Theorem states that for every partition $P : [\mathbf{N}]^e \to r$ [1]) there exists an infinite subset X of **N** such that $P \mid [X]^e$ is constant. In these circumstances one says that X is homogeneous for the partition P. This set-theoretic theorem has various combinatorial consequences which are formalizable in elementary arithmetic. One such immediate consequence which we shall prove independent of the Peano axioms is the following.

Proposition 1. *Let $P : [\mathbf{N}]^e \to r$ be a primitive recursive partition. For every natural number k there exists a finite subset X of* **N**, *with $\# X \geqslant k$ and $\# X \geqslant 2^{2^{\min X}}$, which is homogeneous for the partition P.*

In order to apply Theorem 2 we require the construction of a set which is simultaneously homogeneous for several partitions. This is easily done by the infinite Ramsey Theorem. Suppose $P_1 : [\mathbf{N}]^{e_1} \to r_1$ and $P_2 : [\mathbf{N}]^{e_2} \to r_2$ are two partitions. Let X_1 be an infinite subset of **N** homogeneous for P_1. Then $P_2 \mid [X]^{e_2}$ is a partition of $[X_1]^{e_2}$, and hence there is is an infinite subset X_2 of X_1 which is homogeneous for P_2 (as well as P_1). This proof extends immediately to finitely many partitions. A direct consequence is the following generalization of Proposition 1.

Proposition 2. *Let $P_i : [\mathbf{N}]^{e_i} \to r_i$, $i \leqslant i \leqslant n$ be a set of primitive recursive partitions. For every natural number k there exists a finite subset X of* **N** *with $\# X \geqslant k$ and $\# X \geqslant 2^{2^{\min X}}$, which is simultaneously homogeneous for all the partitions $P_1, \ldots, P_n$.*

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[1]) We identify the number r with the set of all natural numbers $< r$.

Proposition 2 may be expressed by a $\prod_2^0$ formula. First it is clear that we can construct a $\sum_1^0$-formula ϕ_i that expresses the properties that

1. $P_i : [\mathrm{N}]^{e\,i} \to r_i$ is a primitive recursive partition
2. $z_1 < z_2 < \ldots < z_{n_k}$
3. $\{ z_1, \ldots, z_{n_k} \}$ is homogeneous for P_i
4. $k \leqslant n_k$
5. $2^{2^{z_1}} \leqslant n_k$

Proposition 2 asserts that for every k

$$\mathbf{N} \models \exists z_1 \ldots \exists z_{n_k} \bigwedge_{i \leq k} \phi_i \, .$$

V. Construction of the Model

We now have all the ingredients at hand to construct a non-standard model of Peano arithmetic, and we have only to assemble them according to the specifications of Section II.

Let P_i be an effective enumeration of all primitive recursive partitions $P_i : [\mathrm{N}]^{e\,i} \to r_i$. By Proposition 2 we have that for every k

$$\mathbf{N} \models \exists z_1 \ldots \exists z_{n_k} \bigwedge_{i \leq k} \phi_i$$

where ϕ_i is the $\sum_1^0$-formula of Section IV expressing the conditions (1)-(5) satisfied by the partition P_i.

Following the prescription given in Section III we let a_{kn_k} be the smallest number such that $a_{k1}, \ldots, a_{kn_k}$ is an increasing sequence satisfying the formula $\bigwedge_{j \leq k} \phi_j$. Now we define the functions h_j by

$$h_0\,(k) \;=\; n_k \quad \text{for every } k$$

and for $j > 0$

$$h_j\,(k) \;=\; \begin{cases} a_{kj} & \text{for } j \leqslant n_k \\ h_{j-1}\,(k)^2 & \text{for } j > n_k \, . \end{cases}$$

Let $\mathscr{F} \;=\; \{ f \,|\, f \leqslant h_j \}$.

Since $\mathbf{1} \leqslant h_o$ the function $\mathbf{1}$ is automatically in $\mathscr{F}$.

By Theorem 2 the sequence $\{ h_j \}$ satisfies $\bigwedge_{j < \infty} \phi_j$ in $\mathscr{F}/D$. We now prove that this implies that the sequence $\{ h_j \}$ satisfies the Stability and