

# IV. Ramsey-type Theorems

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then we can construct the sequence  $\{h_i\}$  and the set  $\mathcal{F}$  to satisfy the Stability and Closure Conditions via Theorem 2.

We could now proceed to show that the above condition is indeed satisfied in  $\mathbf{N}$  and thus construct a non-standard model of Peano arithmetic. However, our goal is the construction of a mathematically perspicuous model which is independent of the logical formulas. The functions  $\{h_i\}$  given by the above condition require the logical calculus in their definition. Accordingly, we shall consider a larger class  $\mathcal{F}$  of functions than those defined above, which we shall construct independently of logical formulas. This class will be constructed from combinatorial principles derived from Ramsey's Partition Theorem.

#### IV. RAMSEY-TYPE THEOREMS

The infinite Ramsey Theorem states that for every partition  $P : [\mathbf{N}]^e \rightarrow r$  ( $r \geq 1$ ) there exists an infinite subset  $X$  of  $\mathbf{N}$  such that  $P \upharpoonright [X]^e$  is constant. In these circumstances one says that  $X$  is homogeneous for the partition  $P$ . This set-theoretic theorem has various combinatorial consequences which are formalizable in elementary arithmetic. One such immediate consequence which we shall prove independent of the Peano axioms is the following.

**PROPOSITION 1.** *Let  $P : [\mathbf{N}]^e \rightarrow r$  be a primitive recursive partition. For every natural number  $k$  there exists a finite subset  $X$  of  $\mathbf{N}$ , with  $\# X \geq k$  and  $\# X \geq 2^{2^{\min X}}$ , which is homogeneous for the partition  $P$ .*

In order to apply Theorem 2 we require the construction of a set which is simultaneously homogeneous for several partitions. This is easily done by the infinite Ramsey Theorem. Suppose  $P_1 : [\mathbf{N}]^{e_1} \rightarrow r_1$  and  $P_2 : [\mathbf{N}]^{e_2} \rightarrow r_2$  are two partitions. Let  $X_1$  be an infinite subset of  $\mathbf{N}$  homogeneous for  $P_1$ . Then  $P_2 \upharpoonright [X_1]^{e_2}$  is a partition of  $[X_1]^{e_2}$ , and hence there is an infinite subset  $X_2$  of  $X_1$  which is homogeneous for  $P_2$  (as well as  $P_1$ ). This proof extends immediately to finitely many partitions. A direct consequence is the following generalization of Proposition 1.

**PROPOSITION 2.** *Let  $P_i : [\mathbf{N}]^{e_i} \rightarrow r_i$ ,  $i \leq i \leq n$  be a set of primitive recursive partitions. For every natural number  $k$  there exists a finite subset  $X$  of  $\mathbf{N}$  with  $\# X \geq k$  and  $\# X \geq 2^{2^{\min X}}$ , which is simultaneously homogeneous for all the partitions  $P_1, \dots, P_n$ .*

<sup>1)</sup> We identify the number  $r$  with the set of all natural numbers  $< r$ .

Proposition 2 may be expressed by a  $\prod_2^0$  formula. First it is clear that we can construct a  $\sum_1^0$ -formula  $\phi_i$  that expresses the properties that

1.  $P_i : [\mathbb{N}]^{e_i} \rightarrow r_i$  is a primitive recursive partition
2.  $z_1 < z_2 < \dots < z_{n_k}$
3.  $\{z_1, \dots, z_{n_k}\}$  is homogeneous for  $P_i$
4.  $k \leq n_k$
5.  $2^{2^{z_1}} \leq n_k$

Proposition 2 asserts that for every  $k$

$$\mathbb{N} \models \exists z_1 \dots \exists z_{n_k} \bigwedge_{i \leq k} \phi_i .$$

### V. CONSTRUCTION OF THE MODEL

We now have all the ingredients at hand to construct a non-standard model of Peano arithmetic, and we have only to assemble them according to the specifications of Section II.

Let  $P_i$  be an effective enumeration of all primitive recursive partitions  $P_i : [\mathbb{N}]^{e_i} \rightarrow r_i$ . By Proposition 2 we have that for every  $k$

$$\mathbb{N} \models \exists z_1 \dots \exists z_{n_k} \bigwedge_{i \leq k} \phi_i$$

where  $\phi_i$  is the  $\sum_1^0$ -formula of Section IV expressing the conditions (1)-(5) satisfied by the partition  $P_i$ .

Following the prescription given in Section III we let  $a_{kn_k}$  be the smallest number such that  $a_{k_1}, \dots, a_{kn_k}$  is an increasing sequence satisfying the formula  $\bigwedge_{j \leq k} \phi_j$ . Now we define the functions  $h_j$  by

$$h_0(k) = n_k \quad \text{for every } k$$

and for  $j > 0$

$$h_j(k) = \begin{cases} a_{kj} & \text{for } j \leq n_k \\ h_{j-1}(k)^2 & \text{for } j > n_k . \end{cases}$$

Let  $\mathcal{F} = \{f \mid f \leq h_j\}$ .

Since  $1 \leq h_0$  the function 1 is automatically in  $\mathcal{F}$ .

By Theorem 2 the sequence  $\{h_j\}$  satisfies  $\bigwedge_{j < \infty} \phi_j$  in  $\mathcal{F}/D$ . We now prove that this implies that the sequence  $\{h_j\}$  satisfies the Stability and