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## 1. THE SHEAF REPRESENTATION OF BOOLEAN ALGEBRA EXTENSIONS

Let  $\mathcal{L}$  be any language for first-order predicate logic. Suppose  $X$  is a non-empty set and for every  $p \in X$  we have an  $\mathcal{L}$ -structure  $\mathcal{B}_p = (B_p, \dots)$ ; put  $S = \bigcup_{p \in X} B_p$ . Suppose  $\varphi(x_1 \dots x_n)$  is an  $\mathcal{L}$ -formula,  $u \subseteq X$  and  $f_1, \dots, f_n : u \rightarrow S$  are such that  $f_i(p) \in B_p$  for  $1 \leq i \leq n$  and  $p \in u$ . Then let

$$\|\varphi[f_1 \dots f_n]\| = \{p \in u \mid \mathcal{B}_p \models \varphi[f_1(p) \dots f_n(p)]\}.$$

We may think of  $\|\varphi[f_1 \dots f_n]\| \subseteq X$  as being a (Boolean) truth value of  $\varphi[f_1 \dots f_n]$  in the power set of  $X$ .

A sheaf of  $\mathcal{L}$ -structures is a sequence

$$\mathcal{S} = (S, \pi, X, \mu)$$

such that a)  $S$  and  $X$  are topological spaces and  $\pi : S \rightarrow X$  is a continuous open local homeomorphism from  $S$  onto  $X$ , b)  $\mu$  is a function assigning to each  $p \in X$  an  $\mathcal{L}$ -structure  $\mathcal{B}_p = (B_p, \dots)$  such that the  $B_p$  are pairwise disjoint,  $S = \bigcup_{p \in X} B_p$  and  $\pi(s) = p$  iff  $s \in B_p$ , c) for every open subset  $u$  of  $X$  and continuous  $f_1, \dots, f_n : u \rightarrow S$  satisfying  $f_i(p) \in B_p$  for  $p \in u$  and every atomic  $\mathcal{L}$ -formula  $\varphi(x_1 \dots x_n)$ ,  $\|\varphi[f_1 \dots f_n]\|$  is an open subset of  $u$ .

The  $\mathcal{L}$ -structure  $\mathcal{B}_p$  is called the stalk of  $\mathcal{S}$  over  $p$ . — Let, if  $\mathcal{S}$  is a sheaf of  $\mathcal{L}$ -structures,  $\Gamma(\mathcal{S})$  be the set of all continuous functions  $f : X \rightarrow S$  satisfying  $f(p) \in B_p$  for  $p \in X$  (the set of “global sections” of  $\mathcal{S}$ ). So  $\Gamma(\mathcal{S})$  is, if non-empty, (the underlying set of) a substructure of the product structure  $\prod_{p \in X} \mathcal{B}_p$ , hence an  $\mathcal{L}$ -structure.

For the rest of the paper, let  $\mathcal{L} = \{+, \cdot, -, 0, 1, U\}$  where  $U$  is a unary predicate. We indicate how, for a given  $BA$  extension  $(B, A)$ ,  $B$  may be represented by  $\Gamma(\mathcal{S})$  where  $\mathcal{S}$  is a sheaf of  $\mathcal{L}$ -structures over a Boolean space. We omit most of the proofs which are easy and entirely analogous to well-known representation theorems for lattices over Boolean spaces. Let  $X$  be the Stone space of  $A$ , i.e. the set of all ultrafilters of  $A$  with the usual topology. For  $p \in X$ , let  $\langle p \rangle$  be the filter of  $B$  generated by  $p$ . Let  $\pi_p : B \rightarrow B/\langle p \rangle = B_p$  be the canonical epimorphism. So  $B_p$  is a  $BA$  with at least two elements. For  $p, q \in X$  and  $p \neq q$ ,  $B_p$  and  $B_q$  are disjoint. Let  $S = \bigcup_{p \in X} B_p$  and  $\pi : S \rightarrow X$  be defined as stated in b) above. Let, for  $p \in X$ ,  $\mu(p)$  be the  $\mathcal{L}$ -structure  $(B_p, \dots, \{0, 1\})$ . For  $u \subseteq X$  open and  $b \in B$ , let  $M_{ub} = \{\pi_p(b) \mid p \in u\}$ . The set of these  $M_{ub}$  constitutes a base

for a topology of  $S$ , and this makes  $\mathcal{S} = (S, \pi, X, \mu)$  a sheaf of  $\mathcal{L}$ -structures. Furthermore, for  $b \in B$ ,  $\sigma_b : X \rightarrow S$  defined by  $\sigma_b(p) = \pi_p(b)$  is a global section of  $\mathcal{S}$  and

$$\left. \begin{array}{l} \sigma : B \rightarrow \Gamma(\mathcal{S}) \\ b \mapsto \sigma_b \end{array} \right\}$$

is an isomorphism from  $B$  onto  $\Gamma(\mathcal{S})$ . We shall now identify  $B$  with  $\Gamma(\mathcal{S})$ ; so every  $b \in B$  is a function from  $X$  to  $S$ . This identifies  $A$  with those  $b \in B$  such that for every  $p \in X$   $b(p) = 0$  or  $b(p) = 1$ , i.e. with those  $b \in B$  satisfying  $\|U(b)\| = X$ . Let  $C$  be the  $BA$  of clopen subsets of  $X$  and  $e(c)$  the characteristic function of  $c$  for  $c \in C$ . Thus  $e$  is an isomorphism from  $C$  onto  $A \subseteq B$ .

In the rest of this section, we show that the property of being a Hausdorff sheaf for  $\mathcal{S}$  is equivalent to a property of the extension  $(B, A)$  which reflects, in a way which is first-order expressible in  $\mathcal{L}$ , completeness of the embedding of  $A$  into  $B$ . Recall that, for a sheaf  $\mathcal{S}$  over a Boolean space  $X$ ,  $S$  is a  $T_2$ -space iff, for any  $f, g \in \Gamma(\mathcal{S})$ ,  $\|f = g\|$  is a clopen subset of  $X$ ;  $\mathcal{S}$  is then said to be a Hausdorff sheaf. Call  $A$  relatively complete in  $B$  if, for every  $b \in B$ , there is a largest element  $a \in A$  such that  $a \leq b$ , equivalently: for  $b \in B$ , there is a largest  $a \in A$  such that  $a \cdot b = 0$  or: for  $b \in B$ , there is a smallest  $a \in A$  such that  $b \leq a$ .

1.1. PROPOSITION.  $\mathcal{S}$  is a Hausdorff sheaf iff  $A$  is relatively complete in  $B$ .

*Proof.* Suppose  $\mathcal{S}$  is Hausdorff and  $b \in B$ . Let  $d \in B$  such that  $d(p) = 0$  for every  $p \in X$ , let  $c = \|b = d\|$  and  $a = e(c)$ . Then  $a$  is the largest element of  $A$  satisfying  $a \cdot b = 0$ .

Conversely, let  $A$  be relatively complete in  $B$  and suppose  $f, g \in B$ . Let  $a$  be the largest element of  $A$  such that  $a \leq f \cdot g + -f \cdot -g$ . Let  $c \in C$  such that  $a = e(c)$ . Then  $\|f = g\| = c$  is a clopen subset of  $X$ .

1.2. REMARK. Let  $A$  be relatively complete in  $B$ . Then the inclusion map from  $A$  to  $B$  is a complete homomorphism.

*Proof.* Suppose  $M$  is a subset of  $A$  having a supremum  $a$  in  $A$ . We show that  $a$  is the supremum of  $M$  in  $B$ . Clearly,  $a$  is an upper bound for  $M$  in  $B$ . Suppose that  $b$  is another upper bound for  $M$  in  $B$ . Let  $\alpha \in A$  be the largest element of  $A$  such that  $\alpha \leq b$ . For every  $m \in M \subseteq A$ , we have  $m \leq b$ , hence  $m \leq \alpha$  and  $a \leq \alpha \leq b$ .

The following facts are trivial:

1.3. REMARK. *a)* Let  $A$  and the inclusion map from  $A$  to  $B$  be complete. Then  $A$  is relatively complete in  $B$ .

*b)* Suppose  $A$  is relatively complete in  $B$  and  $B$  is complete. Then  $A$  is complete.

## 2. RELATIVE AUTOMORPHISMS OF FINITE EXTENSIONS

We first give an internal description of a finite extension  $(B, A)$  where  $B = A(u_1 \dots u_n)$  and  $n \in \omega$ . We shall always assume that  $u_1, \dots, u_n$  are the atoms of the subalgebra of  $B$  generated by  $u_1, \dots, u_n$ ; i.e. that they are non-zero, pairwise disjoint and  $u_1 + \dots + u_n = 1$ . Let  $I_r = \{a \in A \mid a \cdot u_r = 0\}$  for  $1 \leq r \leq n$ . Clearly, each  $I_r$  is a proper ideal of  $A$  and  $I_1 \cap \dots \cap I_n = \{0\}$ . The family  $(I_r \mid 1 \leq r \leq n)$  completely characterizes the extension  $(B, A)$ :

2.1. REMARK. Suppose  $C = A(v_1 \dots v_n)$  is a finite extension of  $A$  where  $v_1, \dots, v_n$  are pairwise disjoint and  $1 = v_1 + \dots + v_n$ . Let  $B = A(u_1 \dots u_n)$  be as above. There is an isomorphism  $g$  from  $B$  onto  $C$  satisfying  $g(a) = a$  for  $a \in A$  and  $g(u_r) = v_r$  iff, for each  $r$ ,  $\{a \in A \mid a \cdot v_r = 0\} = I_r$ .

*Proof.* By Theorem 12.4 in [7].

2.2. REMARK.  $A$  is relatively complete in  $B = A(u_1 \dots u_n)$  iff, for each  $r$ ,  $I_r$  is a principal ideal.

*Proof.* The only-if part follows by the definition of relative completeness. Now suppose  $\alpha_r \in A$  generates  $I_r$ ; let  $b \in B$  and  $I = \{a \in A \mid a \cdot b = 0\}$ . There are  $a_1, \dots, a_n \in A$  such that  $b = a_1 \cdot u_1 + \dots + a_n \cdot u_n$ . It follows that  $I$  is the principal ideal generated by  $\alpha = (-a_1 + \alpha_1) \cdot \dots \cdot (-a_n + \alpha_n)$ .

Conversely, given any family  $(I_r \mid 1 \leq r \leq n)$  of proper ideals in  $A$  satisfying  $I_1 \cap \dots \cap I_n = \{0\}$ , there is an extension  $A(u_1 \dots u_n)$  of  $A$  such that  $I_r = \{a \in A \mid a \cdot u_r = 0\}$ : let  $D = A(x_1 \dots x_n)$  be the free product of  $A$  and a finite BA with atoms  $x_1, \dots, x_n$ . Let

$$K = \{i_1 \cdot x_1 + \dots + i_n \cdot x_n \mid i_1 \in I_1, \dots, i_n \in I_n\}.$$

$K$  is an ideal of  $D$ ; the canonical epimorphism  $\pi$  from  $D$  onto  $B = D/K$  is one-one on  $A$ , and for  $a \in A$ ,  $\pi(a) \cdot u_r = 0$  iff  $a \in I_r$  where  $u_r = \pi(x_r)$ . Now identify  $A$  with the subalgebra  $\pi(A)$  of  $B$ .

For the rest of this section we think, as in section 1, of  $B$  as being the set of global sections of a sheaf  $\mathcal{S} = (S, \pi, X, \mu)$  of Boolean algebras over a