# 3. Truth values in for statements about (B, A)

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## 3. Truth values in $A$ for statements about $(B, A)$

For the rest of this paper, let $\mathscr{L}_{BA} = \{+, \cdot, -, 0, 1\}$ the language of $BA$s and $\mathscr{L} = \mathscr{L}_{BA} \cup \{U\}$. Let $T_{BAU}$ be the theory in $\mathscr{L}$ such that the models of $T_{BAU}$ have the form $(B, +, \cdot, -, 0, 1, A)$ where $(B, ...)$ is a $BA$ and $A$ is a subalgebra of $B$. We abbreviate a model $(B, ..., A)$ of $T_{BAU}$ by $\mathscr{M} = (B, A)$. We assume the construction and notations of section 1. For each $\mathscr{L}$-formula $\varphi(x_1 \dots x_n)$ and $b_1, ..., b_n \in B$, we defined

$$\| \varphi [b_1 \dots b_n] \| = \{ p \in X \mid B_p \models \varphi [b_1 (p) \dots b_n (p)] \}$$

where $B_p$ abbreviates $(B_p, 2)$ and 2 is the two-element $BA$. Our first claim is that if $c = \| \varphi [b_1 \dots b_n] \|$ is a clopen subset of $X$ for every $\varphi$, then $e(c) \in A$ is first-order definable in $\mathscr{M} = (B, A)$ from the parameters $b_1, ..., b_n \in B$:

3.1. Lemma. There is an effective procedure assigning to each formula $\varphi(x_1 \dots x_n)$ of $\mathscr{L}$ a formula $s_\varphi(yx_1 \dots x_n)$ of $\mathscr{L}$ (where $y$ is a variable not occurring in $\varphi$) such that for $\mathscr{M} \models T_{BAU}$, properties (i) and (ii) are equivalent and (ii) implies (iii):

(i) $\| \varphi [b_1 \dots b_n] \|$ is clopen for every $\varphi(x_1 \dots x_n)$ in $\mathscr{L}$ and $b_1, ..., b_n \in B$;

(ii) $\mathscr{M} \models \forall x_1 \dots \forall x_n \exists y \, s_\varphi(yx_1 \dots x_n)$ for every $\varphi(x_1 \dots x_n)$ in $\mathscr{L}$;

(iii) if $b_1, ..., b_n \in B$, then $a = e(c)$ where $c = \| \varphi [b_1 \dots b_n] \|$ is the unique element $b$ of $B$ such that $\mathscr{M} \models s_\varphi [bb_1 \dots b_n]$.

*Proof.* The inductive definition of $s_\varphi$ will show that (i) is equivalent to (ii) and (i) implies (iii), the interesting cases being $\varphi$ atomic or $\varphi$ existential. In both cases the fact that $\| \varphi [...] \|$ is clopen will be expressed by stating "$a\,(= e(\| \varphi [...] \|)$ is the largest element of $A$ such that $e^{-1}(a) \subseteq \| \varphi [...] \|$". This includes, if $\varphi$ has the form $\exists x \psi$, the maximum principle for the Boolean valuation

$$\psi, b_1 \dots b_n \to \| \psi [b_1 \dots b_n] \|$$

of $\mathscr{M}$ in $C$: there is some $b \in B$ such that

$$\| \psi [b' b_1 \dots b_n] \| \leqslant \| \psi [bb_1 \dots b_n] \|$$

for every $b' \in B$, and hence $\| \psi [bb_1 \dots b_n] \| = \| \exists x \psi [xb_1 \dots b_n] \|$. We now proceed to define the formulas $s_\varphi$.

*a)* Suppose $\varphi$ is an atomic formula of $\mathscr{L}_{BA}$, i.e. $\varphi$ has the form $t_1(x_1 \dots x_n) = t_2(x_1 \dots x_n)$ where $t_1, t_2$ are terms in $\mathscr{L}_{BA}$. Let $s_\varphi(yx_1 \dots x_n)$ be the formula

$$U(y) \wedge y \cdot t_1 = y \cdot t_2 \wedge \forall y' (U(y') \wedge y' \cdot t_1 = y' \, t_2 \to y' \leqslant y) .$$

*b)* Suppose $\varphi$ has the form $U(t(x_1 \dots x_n))$ where $t$ is a term in $\mathscr{L}_{BA}$. Let $\psi$, $\chi$ be the atomic $\mathscr{L}_{BA}$-formulas "$t = 1$" resp. "$t = 0$". Let $s_\varphi$ be the formula

$$\exists y_1 \exists y_2 [y = y_1 + y_2 \wedge s_\psi(y_1 x_1 \dots x_n) \wedge s_\chi(y_2 x_1 \dots x_n)] .$$

*c)* Suppose $\varphi$ has the form $\neg \psi(x_1 \dots x_n)$. Let $s_\varphi$ be the formula

$$\exists y_1 [y = -y_1 \wedge s_\psi(y_1 x_1 \dots x_n)] .$$

*d)* Suppose $\varphi$ has the form $\psi(x_1 \dots x_n) \vee \chi(x_1 \dots x_n)$. Let $s_\varphi$ be the formula

$$\exists y_1 \exists y_2 [y = y_1 + y_2 \wedge s_\psi(y_1 x_1 \dots x_n) \wedge s_\chi(y_2 x_1 \dots x_n)] .$$

*e)* Suppose $\varphi$ has the form $\exists x \, \psi(x x_1 \dots x_n)$. Let $s_\varphi$ be the formula

$$\exists x s_\psi(y x x_1 \dots x_n) \wedge \forall x' \forall y' [s_\psi(y' x' x_1 \dots x_n) \to y' \leqslant y] .$$

Let $\sigma$ be the $\mathscr{L}_{BA}$-formula stating that the supremum of the atoms of a *BA* exists; $\sigma^U$ is the relativization of $\sigma$ to the one-place predicate $U$ of $\mathscr{L}$. The models of $T_{BA} \cup \{\sigma\}$ are called separated *BA*s in [3]. Let $T$ be the $\mathscr{L}$-theory

$$T = T_{BAU} \cup \{\forall x_1 \dots \forall x_n \exists y \, s_\varphi(y x_1 \dots x_n) \mid \varphi(x_1 \dots x_n) \text{ in } \mathscr{L}\} \cup \{\sigma^U, s_\sigma(1)\} .$$

The last two axioms of $T$ express, for a model $\mathscr{M} = (B, A)$ of $T_{BAU}$, that $A$ and each stalk $B_p$ are separated *BA*s. Let **K** be the class of $\mathscr{L}$-structures $\mathscr{M} = (B, A)$ where $B$ is a *cBA* and $A$ is relatively complete in $B$. We shall prove in section 4 that $T$ is an axiomatization of the first-order theory of **K**. The easy part of this is:

3.2. Theorem. *Each structure $\mathscr{M}$ in* **K** *is a model of $T$.*

*Proof.* Let $\mathscr{M} = (B, A) \in \mathbf{K}$, i.e. $B$ is complete and $A$ is relatively complete in $B$. Hence $\mathscr{M} \models T_{BAU}$ and $A$ is a separated *BA*. By 1.1, $\| \varphi[b_1 \dots b_n] \|$ is clopen for every atomic formula $\varphi$ of $\mathscr{L}$ and arbitrary $b_1, \dots, b_n \in B$. If $\| \varphi[b_1 \dots b_n] \|$ and $\| [\psi[b_1 \dots b_n] \|$ are clopen subsets of $X$, so are $\| \neg \varphi[b_1 \dots b_n] \|$ and $\| (\varphi \vee \psi)[b_1 \dots b_n] \|$. Hence we assume that $\varphi$

has the form $\exists x\, \psi\,(xx_1 \ldots x_n)$ and that $\| \psi\,[bb_1 \ldots b_n] \|$ is clopen for fixed $b_1, \ldots, b_n \in B$ and arbitrary $b \in B$. For the rest of the proof, we omit the parameters $b_1 \ldots, b_n$. Let

$$u = \cup \left\{ \| \psi\,[\beta] \| \,\middle|\, \beta \in B \right\}.$$

By our inductive assumption, $u$ is an open subset of $X$. Choose, by Zorn's lemma, a maximal family $F = \left\{ (b_i, c_i) \,\middle|\, i \in I \right\}$ such that $b_i \in B$, $c_i$ is a clopen subset of $u$, $c_i \subseteq \| \psi\,[b_i] \|$, $i \neq j$ implies $c_i \cap c_j = \phi$. It follows that $c$, the closure of $\bigcup_{i \in I} c_i$, includes $u$ (by maximality of $F$). $A$ is a $cBA$, hence $X$ is extremally disconnected and $c$ is clopen. By completeness of $B$, there is some $b \in B$ such that $b \cdot e\,(c_i) = b_i$ for $i \in I$. Thus, for $i \in I$, $c_i \subseteq \| \psi\,[b] \|$. So, for $\beta \in B$, $\| \psi\,[\beta] \| \subseteq u \subseteq c \subseteq \| \psi\,[b] \| = \| \exists x \psi\,(x) \|$.

Finally we show that $B_p$ is separated for each $p \in X$. Let $\alpha\,(x)$ be the $\mathscr{L}_{BA}$-formula stating that $x$ is an atom and let $\beta\,(x)$, $\gamma\,(x)$ be the $\mathscr{L}_{BA}$-formulas $\alpha\,(x) \vee x = 0$ resp. $\forall y\, (\alpha\,(y) \rightarrow y \leqslant x)$. Put $M = \{ f \in B \mid \| \beta\,[f] \| = 1 \}$ and let $b$ be the supremum of $M$ in $B$. We show that $b\,(p)$ is, for each $p \in X$, the supremum of the atoms of $B_p$.

First suppose $s \in B_p$ is an atom of $B_p$. There is some $f \in M$ such that $f\,(p) = s$ (note that $\| \alpha\,[f] \|$ is clopen for each $f \in B$). So $f \leqslant b$ and $s = f\,(p) \leqslant b\,(p)$. — On the other hand, suppose $t \in B_p$ and $s \leqslant t$ for every atom $s$ of $B_p$. Choose $g \in B$ such that $g\,(p) = t$. Then $p \in c = \| \gamma\,[g] \|$. For $f \in M$, $e\,(c) \cdot f \leqslant g$, since $q \in c$ implies that $f\,(q)$ is zero or an atom of $B_q$ and thus $f\,(q) \leqslant g\,(q)$. By the definition of $b$, $e\,(c) \cdot b \leqslant g$. This implies (by $p \in c$) $b\,(p) \leqslant g\,(p) = t$.

## 4. Decidability and completions of $Th\,(\mathbf{K})$

Call $T_{sBA} = T_{BA} \cup \{ \sigma \}$ the theory of separated $BAs$, where $T_{BA}$ is the theory of $BAs$ and $\sigma$ was defined in section 3. We give a short review of the completions of $T_{sBA}$. Let, for $n \in \omega$, $\varphi_n$ be the $\mathscr{L}_{BA}$-sentence stating that there are exactly $n$ atoms and $\psi$ the $\mathscr{L}_{BA}$-sentence stating that there is a non-zero atomless element. Let $\chi_n = \neg\, (\varphi_0 \vee \ldots \vee \varphi_{n-1})$; so $\chi_n$ says that there are at least $n$ atoms. Define, for $n \in \omega + 1$ and $i \in 2 = \{0, 1\}$, an $\mathscr{L}_{BA}$-theory $T_{ni}$ by

$$T_{n0} = T_{sBA} \cup \{ \varphi_n, \neg\, \psi \}$$
$$T_{n1} = T_{sBA} \cup \{ \varphi_n, \psi \}$$