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ON POLYLOGARITHMS, HURWITZ ZETA FUNCTIONS, AND THE KUBERT IDENTITIES

by John MILNOR

§1. INTRODUCTION

D. Kubert [12] has studied functions $f(x)$, where x varies over $\mathbf{Q}/\mathbf{Z}$ or $\mathbf{R}/\mathbf{Z}$, which satisfy the identity

$$(*_s) \qquad f(x) = m^{s-1} \textstyle\sum_{k=0}^{m-1} f\big((x+k)/m\big)$$

for every positive integer m. (See also Lang [16-18], as well as Kubert and Lang [13-15].) Here s is some fixed parameter. Note that $(x+k)/m$ varies precisely over all solutions y to the equation $my = x$ in the group $\mathbf{Q}/\mathbf{Z}$ or $\mathbf{R}/\mathbf{Z}$. However, the equation is set up so that it also makes sense for x in the interval $(0, 1)$ or $(0, \infty)$. Evidently it would suffice to assume the equation $(*_s)$ for prime values of m.

Classical examples of such functions are provided by the uniformly convergent Fourier series $l_s(x) = \sum_{n=1}^{\infty} e^{2\pi i n x}/n^s$ for $x \in \mathbf{R}/\mathbf{Z}$ and $Re(s) > 1$, the Hurwitz function

$$\zeta_{1-s}(x) = x^{s-1} + (x+1)^{s-1} + \ldots$$

for $0 < x$ and $Re(s) < 0$, and by the Bernoulli polynomial $\beta_s(x)$ of degree s for $s = 0, 1, 2, 3, \ldots$. See §2.

For each complex constant s, it is shown in §3 that there are exactly two linearly independent functions, defined and continuous on the open interval $(0, 1)$, which satisfy these Kubert identities $(*_s)$. The two generators may be chosen so that one is even and one is odd under the involution $f(x) \mapsto f(1-x)$. They are then uniquely determined up to a multiplicative constant. Here is a table of examples, for small integer values of s.

	-2	-1	0	1	2
even	$\zeta_3(x) + \zeta_3(1-x)$	$\csc^2 \pi x$	$\beta_0(x) = 1$	$\log(2 \sin \pi x)$	$\beta_2(x) = x^2 - x + \frac{1}{6}$
odd	$\cos \pi x / \sin^3 \pi x$	$\zeta_2(x) - \zeta_2(1-x)$	$\cot \pi x$	$\beta_1(x) = x - \frac{1}{2}$	$\Lambda(\pi x)$

Here the symbol Λ stands for the function

$$\Lambda(\pi x) = -\int_0^{\pi x} \log|2\sin\theta|\,d\theta = \sum_1^\infty \sin(2\pi n x)/2n^2,$$

which is closely related to Lobachevsky's computations of volume in hyperbolic 3-space. Compare Appendix 3.

Section 4 extends such functions from (0, 1) to the circle $\mathbf{R}/\mathbf{Z}$. For any *integer* constant s, §5 computes the universal function

$$u : \mathbf{Q}/\mathbf{Z} \to U_s$$

satisfying the identities $(*_s)$. Here U_s is the abelian group with one generator $u(x)$ for each x in $\mathbf{Q}/\mathbf{Z}$ and with defining relations $(*_s)$.

Section 6 attempts to study the extent to which the continuous Kubert functions of §3 are actually universal, when restricted to $\mathbf{Q}/\mathbf{Z}$. For example, if $f : (0, 1) \to \mathbf{R}$ is the essentially unique even [or odd] continuous function satisfying $(*_s)$, where s is an integer, does every $\mathbf{Q}$-linear relation between the values of f at rational arguments follow from $(*_s)$ together with evenness [or oddness]? The Bernoulli polynomials $\beta_s(x)$ provide obvious counterexamples; but *it is conjectured that these are the only counterexamples.* This question is settled in the relatively easy cases where the values of f on $\mathbf{Q}/\mathbf{Z}$ are known to be algebraic numbers, or logarithms of algebraic numbers.

There are three appendices, one describing a functional equation relating polylogarithms and Hurwitz functions, one describing $\Gamma(x)$ and related functions, and one describing the use of dilogarithms to compute volume in Lobachevsky space.

The author is indebted to conversations with S. Chowla, B. H. Gross, Werner Meyer, and W. Sinnott.

§2. Classical examples

This section describes several well known functions. Since the identities $(*_s)$ are not immediately perspicuous, let me start with some examples where they are clearly satisfied. For any complex constant c the polynomial $t^m - c$ factors as

$$t^m - c = \prod_{b^m = c} (t - b),$$

where b varies over all m-th roots of c. Hence, setting $t = 1$, we see that

$$\log|1 - c| = \sum_{b^m = c} \log|1 - b|.$$