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THE CLEBSCH-GORDAN FORMULAS

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0. INTRODUCTION

The explicit decomposition of tensor products of irreducible representations is of fundamental importance in many applications of representation theory. For finite dimensional representations of the Lie algebra $\mathfrak{sl}_2$ definitive results are contained in the famous Clebsch-Gordan formulas which are constantly and routinely used by physicists in applying the quantum theory of angular momentum. We give in this article a presentation and derivation of equivalent results, Theorems 5.1 and 5.4.

We shall base a study of the representations $\mathrm{Hom}(V, W)$ (rather than $V \otimes W$) for irreducible $\mathfrak{sl}_2$-representations V and W on the analysis of a Weyl algebra $\mathscr{A}$ of polynomial differential operators in two variables. This point of view is one developed in a recent attack on the Clebsch-Gordan problem for $\mathfrak{sl}_3$ [2].

The usefulness of the Weyl algebra in the resolution of the Clebsch-Gordan problem is well-known. For years physicists have worked with it under the name "boson calculus" [1]. One mathematical reference is [3]. Nothing in the present article is new except possibly the arrangement of the proofs which has been made with the benefit of experience gained working with $\mathfrak{sl}_3$. It seems to me that this arrangement has a naturalness and simplicity to recommend it.

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1. SOME REPRESENTATIONS OF $\mathfrak{sl}_2$

Let $V = \mathbf{C}[X, Y]$, the vector space of polynomials in two variables X and Y. For integers m let V_m be the subspace of homogeneous polynomials of degree m, with $V_m = (0)$ for negative m.

Let $SL_2(\mathbf{C})$ act linearly on V as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot X = aX + cY \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot Y = bX + dY \tag{1.1}$$

$$g \cdot X^a Y^b = (g \cdot X)^a (g \cdot Y)^b \qquad \text{for } g \in SL_2(\mathbf{C}). \tag{1.2}$$

Each V_m is an $SL_2(\mathbf{C})$ subrepresentation of V.

By $\mathfrak{sl}_2$ we denote the Lie algebra of 2×2 complex matrices with trace 0. The representation of $SL_2(\mathbf{C})$ on V gives rise, through differentiation, to a representation of $\mathfrak{sl}_2$ on V.

$$L \cdot v = \left.\frac{d}{dt}\right|_{t=0} \exp(tL) \cdot v \qquad \text{for } L \in \mathfrak{sl}_2, v \in V. \tag{1.3}$$

Choose a basis E_+, E_-, H of $\mathfrak{sl}_2$ as follows:

$$E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.4}$$

An easy calculation establishes the following equalities of linear endomorphisms of V.

$$E_+ = X\partial_Y, \qquad E_- = Y\partial_X, \tag{1.5}$$

$$H = X\partial_X - Y\partial_Y. \tag{1.6}$$

From (1.5) and (1.6) one easily deduces that each V_m is an *irreducible* representation of $\mathfrak{sl}_2$ (and of $SL_2(\mathbf{C})$).

We define for integers m, n a representation τ of $\mathfrak{sl}_2$ on $\mathrm{Hom}_{\mathbf{C}}(V_m, V_n)$ by means of formula (1.7).

$$(\tau(L) \cdot T)v = L(Tv) - T(Lv)$$
$$\text{for } L \in \mathfrak{sl}_2, T \in \mathrm{Hom}_{\mathbf{C}}(V_m, V_n), v \in V_m. \tag{1.7}$$

The principal result of this article is the explicit decomposition of the $\mathfrak{sl}_2$-representations $\mathrm{Hom}_{\mathbf{C}}(V_m, V_n)$.

2. THE WEYL ALGEBRA $\mathscr{A}$

Let $\mathscr{A}$ be the subalgebra of $\mathrm{End}_{\mathbf{C}}(V)$ consisting of polynomial differential operators on $V = \mathbf{C}[X, Y]$. The algebra $\mathscr{A}$ is spanned by the elements

$$D(i, j, a, b) = X^i Y^j \partial_X{}^a \partial_Y{}^b. \tag{2.1}$$

The Euler operator J, which acts as scalar multiplication by m on V_m, lies in $\mathscr{A}$.

$$J = X\partial_X + Y\partial_Y . \tag{2.2}$$

The next lemma assures us that $\mathscr{A}$ is large enough for the study of all spaces $\mathrm{Hom}_{\mathbf{C}}(V_m, V_n)$.

LEMMA 2.3. *Let U be a finite dimensional vector subspace of V and let $T \in \mathrm{End}_{\mathbf{C}}(U)$. Then there exists an element of $\mathscr{A}$ whose restriction to U equals T.*

Proof: The element $S = X^c Y^d (\partial_X)^a (\partial_Y)^b \prod_{\substack{m=0 \\ m \neq a+b}}^{N} (J-m)$ of $\mathscr{A}$ maps $X^a Y^b$ to a nonzero multiple of $X^c Y^d$ and kills all other monomials of degree at most N. But by enlarging U we may assume that $\mathrm{End}_{\mathbf{C}}(U)$ is spanned by restrictions of elements of the form S. □

We use the inclusion of $\mathfrak{sl}_2$ in $\mathscr{A}$ to define a representation ρ of $\mathfrak{sl}_2$ on $\mathscr{A}$.

$$\rho(L)a = [L, a] \qquad \text{for } L \in \mathfrak{sl}_2,\ a \in \mathscr{A} . \tag{2.4}$$

For integers n let $\mathscr{A}^n$ be the set of T in $\mathscr{A}$ such that $T(V_m) \subset V_{m+n}$ for all m.

This defines a grading of $\mathscr{A}$ which is preserved by the action of $\mathfrak{sl}_2$.

$$\mathscr{A} = \bigoplus_{n \in \mathbf{Z}} \mathscr{A}^n , \qquad \mathscr{A}^m \cdot \mathscr{A}^n \subset \mathscr{A}^{m+n} , \tag{2.5}$$

$$\rho(L)\mathscr{A}^n \subset \mathscr{A}^n \qquad \text{for all } L \in \mathfrak{sl}_2 . \tag{2.6}$$

The algebra $\mathscr{A}$ and representation ρ have been defined just so that the next lemma, which is an immediate consequence of Lemma 2.3, will be true.

LEMMA 2.7. *For each m, n the restriction map*

$$\mathrm{res}: \mathscr{A}^n \to \mathrm{Hom}_{\mathbf{C}}(V_m, V_{m+n})$$

is a surjective homomorphism of $\mathfrak{sl}_2$ representations. □

The method of this paper is to deduce the decomposition of the representations $\mathrm{Hom}_{\mathbf{C}}(V_m, V_{m+n})$ from the decomposition of the representation ρ on $\mathscr{A}$ by means of Lemma 2.7.

3. The theory of the highest weight

Before decomposing the $\mathfrak{sl}_2$-space $\mathscr{A}$ we must review the finite dimensional representation theory of $\mathfrak{sl}_2$.

The *weight vectors* of an $\mathfrak{sl}_2$-representation W are the eigenvectors of H in W. The *weights* of W are the eigenvalues of its nonzero weight vectors.

Every finite dimensional $\mathfrak{sl}_2$-module is spanned by its weight vectors. The weights of such a representation are all integers and are thus ordered by the usual order on $\mathbf{R}$. The largest of a finite set of integral weights is traditionally referred to as the *highest* weight.

Two finite dimensional irreducible $\mathfrak{sl}_2$-representations are isomorphic if and only if they have the same highest weights, which are necessarily nonnegative.

The element $X^a Y^b$ of V is a weight vector of weight a-b. This shows that X^m is a vector of highest weight m in V_m and therefore that the V_m for $m \geqslant 0$ form a set of representatives of the equivalence classes of finite dimensional irreducible $\mathfrak{sl}_2$-representations; which is precisely why we are studying them is this paper.

The last general fact which we will recall without proof is this: every finite dimensional representation of $\mathfrak{sl}_2$ is a direct sum of irreducible representations.

Given a representation W of $\mathfrak{sl}_2$ which is a sum of finite dimensional representations one often wishes to write it explicitly as a direct sum of irreducible representations, that is, of representations isomorphic to the V_m. A method for doing this is provided by the observation that the space of weight vectors of highest weight in V_m is the space annihilated by E_+ and is one dimensional. Thus for each $v \in W$ of weight m such that $E_+ v = 0$, there is a unique $\mathfrak{sl}_2$-homomorphism from V_m to W taking X^m to v. The explicit decomposition of W therefore amounts to the determination of a basis consisting of weight vectors of the kernel of E_+ in W.

4. The decomposition of $\mathscr{A}$

We apply the procedure of the last paragraph to the representation of $\mathfrak{sl}_2$ on $\mathscr{A}$. By definition of ρ the kernel of $\rho(E_+)$ is just the commutant of E_+ in $\mathscr{A}$.

Let $\mathscr{B}$ be the subalgebra of $\mathscr{A}$ generated by X, ∂_Y, and J.

PROPOSITION 4.1. *$\mathscr{B}$ is the commutant of E_+ in $\mathscr{A}$.*

Proof: One easily verifies that E_+ commutes with X, ∂_Y, and J, which shows that $\mathscr{B}$ is contained in the commutant of E_+.

Let U be the $\mathfrak{sl}_2$-subrepresentation of $\mathscr{A}$ generated by $\mathscr{B}$. The considerations of Section 3 show that the inclusion of the commutant of E_+ in $\mathscr{B}$ is equivalent to the assertion that U equals all of $\mathscr{A}$. We proceed to establish that equality.

The algebra $\mathscr{B}$ is spanned as a vector space by the elements

$$J^a X^b(\partial_Y)^c \qquad \text{with } a, b, c \geqslant 0\,. \tag{4.2}$$

We present two calculations.

$$\begin{aligned}&[E_-, J^a X^b(\partial_Y)^{c+1}]\\ &= -(b+c+1)J^a X^b(\partial_Y)^c\partial_X + b(J+1-b)J^a X^{b-1}(\partial_Y)^c\end{aligned} \tag{4.3}$$

$$\begin{aligned}&[E_-, J^a X^{b+1}(\partial_Y)^c]\\ &= (b+1)J^a X^b(\partial_Y)^c Y - cJ^a X^b(\partial_Y)^{c-1}(b+1+X\partial_X)\,.\end{aligned} \tag{4.4}$$

From (4.3) one concludes that $\mathscr{B}\cdot\partial_X \subset U$. From that and (4.4) one concludes that $\mathscr{B}\cdot Y \subset U$.

Because E_- commutes with ∂_X and Y, one has that

$$\rho(E_-)^n(\mathscr{B}\partial_X) = \big(\rho(E_-)^n\mathscr{B}\big)\cdot\partial_X$$

and that

$$\rho(E_-)^n(\mathscr{B}\cdot Y) = \big(\rho(E_-)^n\mathscr{B}\big)\cdot Y\,.$$

Because $V_m = \bigoplus_{n=0}^{\infty} E_-{}^n(\mathbf{C}X^m)$ one knows that $U = \bigoplus_{n=0}^{\infty}\rho(E_-)^n\mathscr{B}$. And thus

$$U\cdot\partial_X \subset U\,, \qquad U\cdot Y \subset U\,. \tag{4.5}$$

Iterating, we have

$$UY^d(\partial_X)^e \subset U \qquad \text{for } d, e, \geqslant 0\,. \tag{4.6}$$

But $\mathscr{A}$ is generated as an algebra by X, Y, ∂_X, and ∂_Y and so (4.2) and (4.6) prove that $U = \mathscr{A}$. □

COROLLARY 4.7. *$\mathscr{A}^0$ is the subalgebra of $\mathscr{A}$ generated by $\mathfrak{sl}_2$ and J.*

Proof: $\mathscr{A}^0$ is the $\mathfrak{sl}_2$-subrepresentation of $\mathscr{A}$ generated by $\mathscr{A}^0 \cap \mathscr{B}$.

$\mathscr{A}^0 \cap \mathscr{B}$ is spanned by the elements (4.2) such that $b = c$, all of which are of the form $J^a E_+{}^b$. $\square$

We remark that the subalgebra of $\mathscr{A}$ generated by $\mathfrak{sl}_2$ is canonically isomorphic to the universal enveloping algebra of $\mathfrak{sl}_2$. The element $J(J+2)$ equals $H^2 + 2(E_+E_- + E_-E_+)$, the Casimir element for $\mathfrak{sl}_2$. Thus $\mathscr{A}^0$ is a little larger than the enveloping algebra of $\mathfrak{sl}_2$.

For integers l, n define $\mathscr{B}\binom{n}{l}$ to be the set of $T \in \mathscr{B} \cap \mathscr{A}^n$ such that $\rho(H)T = lT$.

This defines a grading of $\mathscr{B}$:

$$\mathscr{B} = \bigoplus \mathscr{B}\binom{n}{l}, \qquad \mathscr{B}\binom{n}{l} \cdot \mathscr{B}\binom{n'}{l'} \subset \mathscr{B}\binom{n+n'}{l+l'}. \tag{4.8}$$

The generators of $\mathscr{B}$ fit in as follows:

$$J \in \mathscr{B}\binom{0}{0}, \qquad X \in \mathscr{B}\binom{1}{1} \qquad \partial_Y \in \mathscr{B}\binom{-1}{1}. \tag{4.9}$$

PROPOSITION 4.10. *i)* $\mathscr{B}\binom{0}{0} = \mathbf{C}[J]$.

ii) $\mathscr{B}\binom{n}{l} \neq 0$ *if and only if* $l \geqslant 0, |n| \leqslant l$, *and* $l \equiv n \pmod 2$. *If these conditions are met, then*

$$\mathscr{B}\binom{n}{l} = \mathbf{C}[J] \cdot X^{\frac{l+n}{2}}(\partial_Y)^{\frac{l-n}{2}} \tag{4.11}$$

Proof: Immediate. $\square$

We note that the condition that $\mathscr{B}\binom{n}{l} \neq (0)$ may be rephrased thus: $l \geqslant 0$ and n is a weight of V_l.

5. DECOMPOSITION OF $\mathrm{Hom}(V_m, V_{m+n})$

THEOREM 5.1. *Let* l, m, n *be integers with* $l, m, m+n \geqslant 0$. *There is an* $\mathfrak{sl}_2$*-subrepresentation of* $\mathrm{Hom}_{\mathbf{C}}(V_m, V_{m+n})$ *which is isomorphic to* V_l *if and only if* $|n| \leqslant l, n \equiv l \pmod 2$, *and* $m \geqslant \dfrac{l-n}{2}$.

Moreover, when these conditions are met there is a unique such subrepresentation. A weight vector of weight l *in it is given by*

$$X^{\frac{l+n}{2}}(\partial_Y)^{\frac{l-n}{2}}.$$

Proof: By Lemma 2.7 and the definition of $\mathscr{B}\binom{n}{l}$, a weight vector of weight l of the subrepresentation sought must be the restriction to V_m of an element of $\mathscr{B}\binom{n}{l}$. By Lemma 4.10ii, all such restrictions are scalar multiples of the restriction of $X^{\frac{l+n}{2}}(\partial_Y)^{\frac{l-n}{2}}$ to V_m, which restriction is nonzero only when $m \geqslant \frac{l-n}{2}$. □

It is interesting to observe that the weight l weight vector in $\mathrm{Hom}_{\mathbf{C}}(V_m, V_{m+n})$ given by Theorem 5.1 is "independent" of m.

Finally we want to give formulas for the weight vectors in $\mathrm{Hom}(V_m, V_{m+n})$ of all weights, not just of highest weight.

For integers l, i, j with $l \geqslant 0$ and $0 \leqslant i,j \leqslant l$, define an element $A_l(i,j)$ of $\mathscr{A}$:

$$A_l(i,j) = \sum_{\alpha \leqslant k \leqslant \beta} (-1)^k \binom{l}{i}\binom{i}{k}\binom{l-i}{j-k} X^{l-i-j+k}\, Y^{j-k}(\partial_X)^k(\partial_Y)^{i-k}$$

$$\text{with } \alpha = \sup\{0, i+j-l\} \quad \text{and} \quad \beta = \inf\{i,j\}. \tag{5.2}$$

LEMMA 5.3. $\rho(E_-)^j \binom{l}{i} X^{l-i}(\partial_Y)^i = j!\, A_l(i,j)$.

Proof: By induction on j. Use the formula:

$$[E_-, D(i,j,a,b)] = iD(i-1, j+1, a, b) - bD(i,j,a+1,b-1)$$

with D as in (2.1). □

THEOREM 5.4. *Let* l, m, n *be such that there is a subrepresentation of* $\mathrm{Hom}_{\mathbf{C}}(V_m, V_{m+n})$ *isomorphic to* V_l. *Then an inclusion of representations* $\phi\colon V_l \to \mathrm{Hom}_{\mathbf{C}}(V_m, V_{m+n})$ *may be given by the formula:*

$$\phi(X^{l-j}Y^j) = \frac{1}{\binom{l}{j}} A_l\left(\frac{l-n}{2}, j\right). \tag{5.5}$$

Proof: This depends on (5.3) and the calculation in V_l that

$$E_-{}^j X^l = \frac{l!}{(l-j)!} X^{l-j} Y^j . \qquad \square$$

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