5. How the Perron tree sprouts

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thought, that we shall call *the Perron tree*, has proved to be an extraordinarily fruitful tool for the solution of certain deep problems of recent mathematical analysis.

The result is as follows: Given an arbitrary $\varepsilon > 0$ and an arbitrary triangle ABC of area that we denote by $S(ABC)$, we can divide the triangle ABC into small triangles $T_1, T_2, \dots, T_n$ as Figure 5 shows (i.e. dividing the basis a into a finite number of equal intervals $I_1, I_2, \dots, I_n$) and one can translate appropriately the small triangles $T_1, T_2, \dots, T_n$ parallelly to the basis a in such a way that the area of the union of the translated triangles is less than $\varepsilon S(ABC)$. (See Fig. 6.)

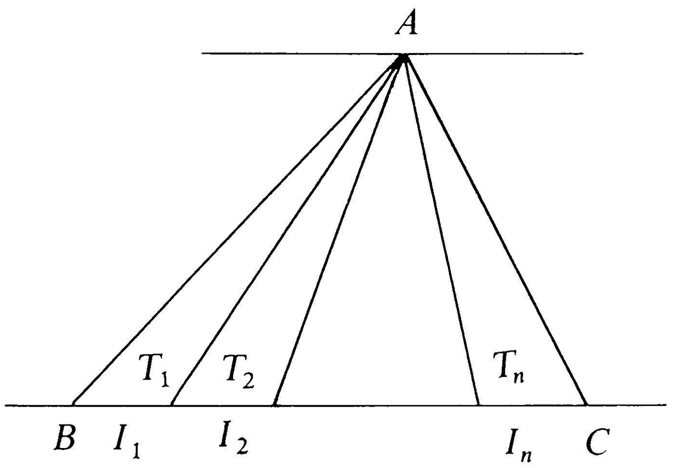

FIGURE 5

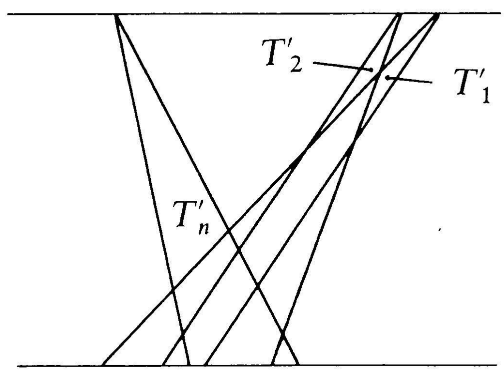

FIGURE 6

5. HOW THE PERRON TREE SPROUTS

Following an idea of Rademacher (1962), the construction of the Perron tree can be easily understood as follows. Let us divide first a triangle T, MNP, of area $S(T)$, into two triangles T_1, T_2, with bases J_1, J_2, of the same length. If we wish to move T_1 and T_2, parallelly to NP so that the shifted triangles cover less area we can do it by pushing T_2 towards T_1 as Figure 7 shows. The area covered by T_1 and T'_2 can be easily measured by elementary geometry and is (see Fig. 7, we take $1/2 < \alpha < 1$)

$$\alpha^2 S(T) + 2(1-\alpha)^2 S(T)$$

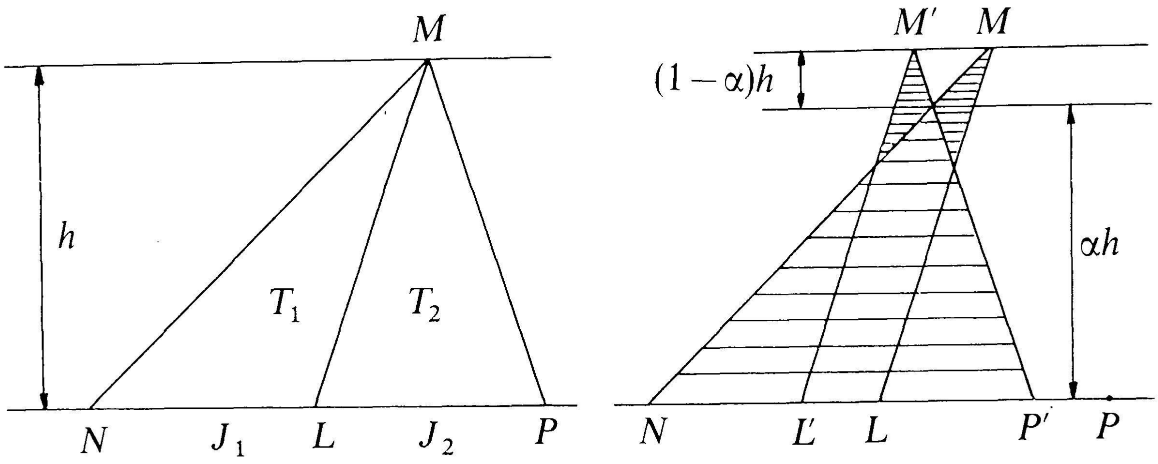

FIGURE 7

If the triangle MNP is divided into four parts, instead of two, as Figure 8 indicates, we can first subject the pair of triangles MNL_1 and ML_1L_2 on the one hand to the above indicated operation with an α, $1/2 < \alpha < 1$, and, on the other hand we can do the same, with the same α, to the other pair of adjacent triangles ML_2L_3, ML_3P. The result is indicated in Figure 8.

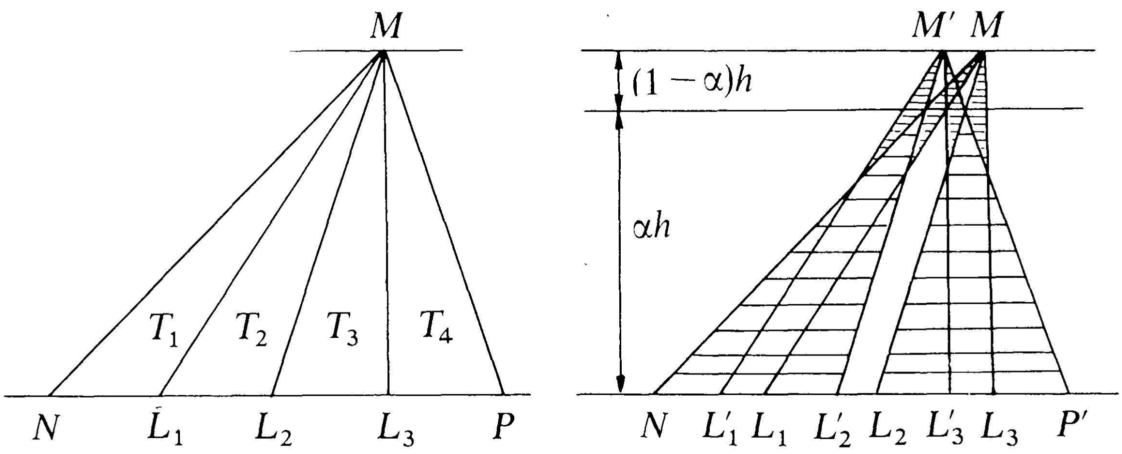

FIGURE 8

It is easy to see that the area of the figure now covered by the so translated triangles is less than

$$(*) \qquad \alpha^2 S(T) + 2(1-\alpha)^2 S(T)$$

If we now shift in a solidary way the figure formed by the union of the two triangles T_3 and T'_4 towards the left until L_2 coincides with L'_2, the new formed figure covered by the four triangles can be considered (see Fig. 9) as consisting of a triangle HNP'' similar to the first one MNP with a similarity ratio α plus four peak triangles that overlap more than before. The area of this figure is therefore less than (*).

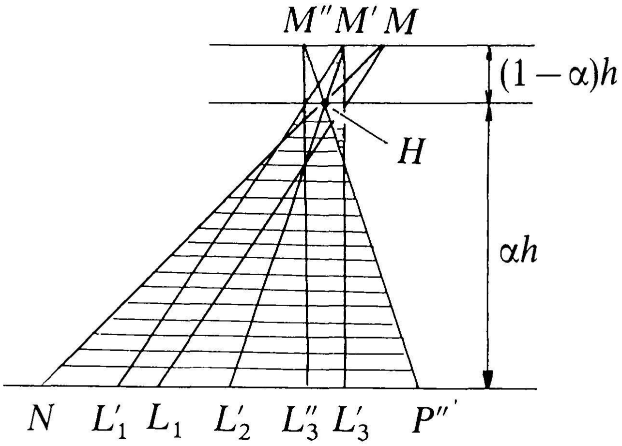

FIGURE 9

In the triangle HNP'' we have the basis divided into equal portions NL_2' and $L_2'P''$ and so we can submit HNP'' to the initial operation, i.e. shifting the right hand triangle towards the left one with the same constant α that measures the magnitude of this shift and shifting thereby solidarily the triangles T_3'' and T_4'' that constitute the right hand portion of the triangle HNP''. The result is shown in Figure 10.

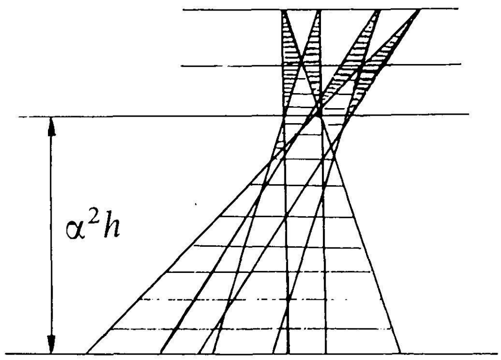

FIGURE 10

The final result is a triangle similar to the initial one with similarity ration α^2, its area therefore being $\alpha^4 S(T)$, plus four peaks that cover an area smaller than

$$2(1-\alpha)^2\alpha^2 S(T) + 2(1-\alpha)^2 S(T)$$

i.e. the total area now covered is not greater than

$$\alpha^4 S(T) + 2(1-\alpha)^2 (1+\alpha^2) S(T)$$

It is now not difficult to realize that if we initiate our process with 2^n equal portions of the basis and we proceed in a similar way, at the end, i.e. after n repetitions of the process consisting of (a) a shift of the right triangle of each pair of adjacent triangles towards the left triangle (shift constant $= \alpha$), and (b) gluing together the resulting figures to compose a triangle similar to the original one with one half the number of divisions on the basis, we obtain a figure with an area not greater than

$$\begin{aligned} &\alpha^{2n} S(T) + 2(1-\alpha)^2 (1+\alpha^2+\ldots+\alpha^{2n-2}) S(T) \\ &\leqslant \alpha^{2n} S(T) + 2(1-\alpha)^2 (1+\alpha^2+\ldots+\alpha^{2n-2}+\ldots) S(T) \\ &= \left(\alpha^{2n} + \frac{2(1-\alpha)^2}{1-\alpha^2}\right) S(T) \leqslant (\alpha^{2n} + 2(1-\alpha)) S(T) \end{aligned}$$

Therefore, given $\varepsilon > 0$, we first choose α, $1/2 < \alpha < 1$ such that $1 - \alpha < \varepsilon/2$ and then n so that $\alpha^{2n} < \varepsilon/2$. In this way we obtain a Perron tree. Its name is justified by the fact that the final figure consists of a trunk (a triangle similar to the initial one with similarity ration α^n) plus many sharp branches that seem to rest on it. Its area is less than $\varepsilon S(T)$. (See Fig. 11.)

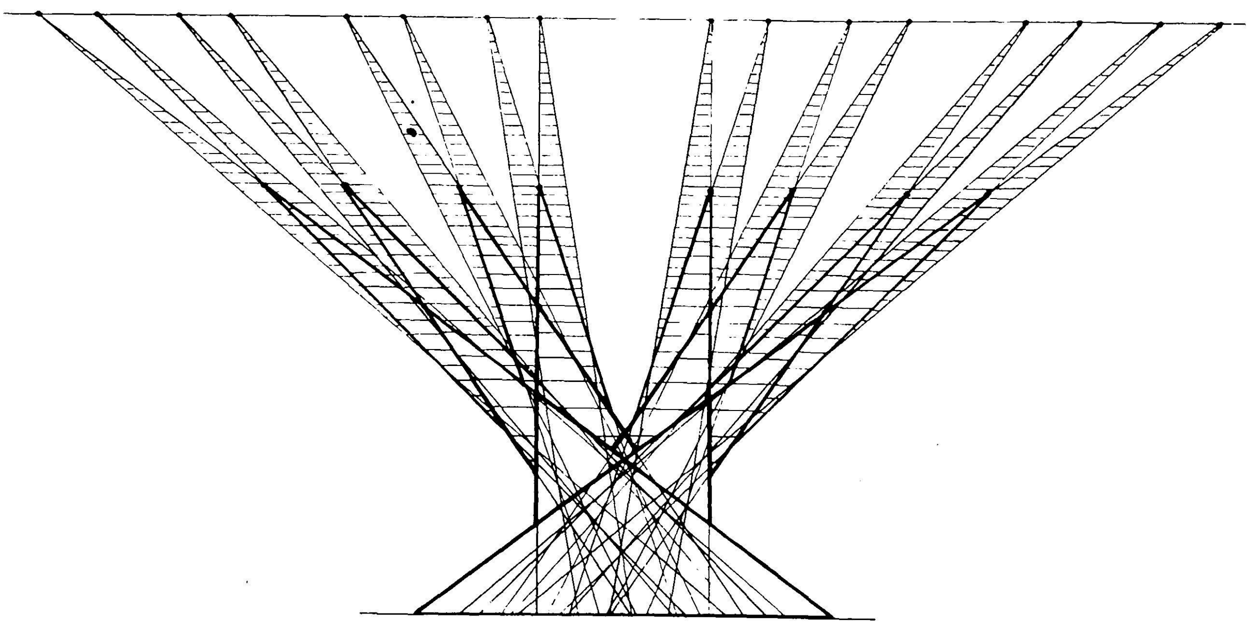

FIGURE 11

The reader interested in more details can consult Guzmán (1975).