7. The construction of the Besocovitch set

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hand side of T'_1. Therefore it can move, using the above construction, covering an area J. Within T'_2, and so within P, it can move to the right hand side of T'_2. From there to the left hand side of T'_3 and so on until it comes to AC, covering area less than $\eta/3$.

It is clear that with three equilateral triangles and three repetitions of this process we can turn the needle around covering area smaller than η.

7. The construction of the Besocovitch set

The Besicovitch set is also easily built starting from the Perron tree by means of the following auxiliary construction:

(**) *Given an arbitrary parallelogram* $ABCD$ *and* $\varepsilon > 0$, *it is possible to construct a finite number of closed parallelograms* $\omega_1, \omega_2, ..., \omega_n$ *so that* (see Fig. 13):

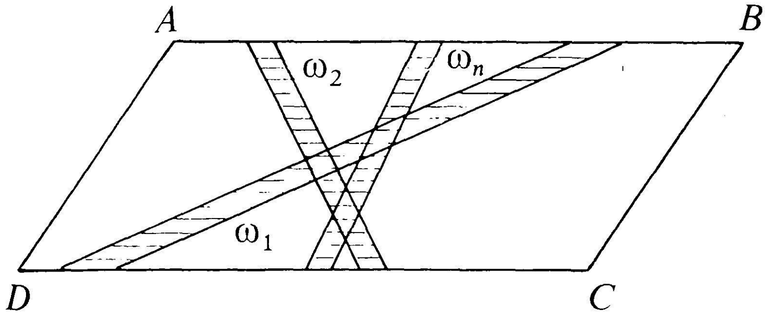

FIGURE 13

(a) *Each one has one basis on* AB *and another one on* CD.

(b) *The area of their union is less than* ε.

(c) *For each segment joining a point of* AB *to another one of* CD *there exists inside some* ω_j *a segment parallel to it of the same length.*

To see this, given $ABCD$ and $\varepsilon > 0$ we first take two strips ω_1 and ω_2 as indicated in Figure 14 such that $S(\omega_1) + S(\omega_2) < \varepsilon/4$. We take now a point L of UV so that LC is parallel to UT. Then we divide VC into intervals with the same length smaller than that of DV and we join L to the extreme points of these intervals. A typical triangle of the ones so obtained is LMN. Let p be

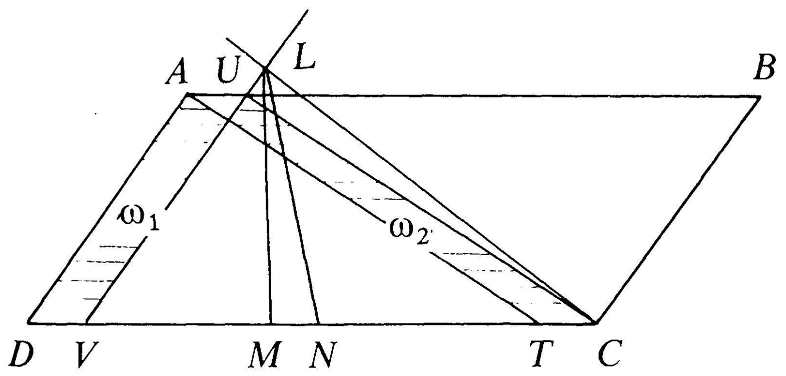

FIGURE 14

the number of such triangles. From LMN we construct a Perron tree with area less than $\varepsilon/4p$ and the same is done with each of the p triangles we have. The union of ω_1, ω_2 and all the small triangles of the p Perron trees has an area less than $\varepsilon/4 + p\,\varepsilon/4p = \varepsilon/2$. One of these small triangles composing one of the Perron trees has the situation indicated in Figure 15 (it has been enlarged to make the figure more easily understandable).

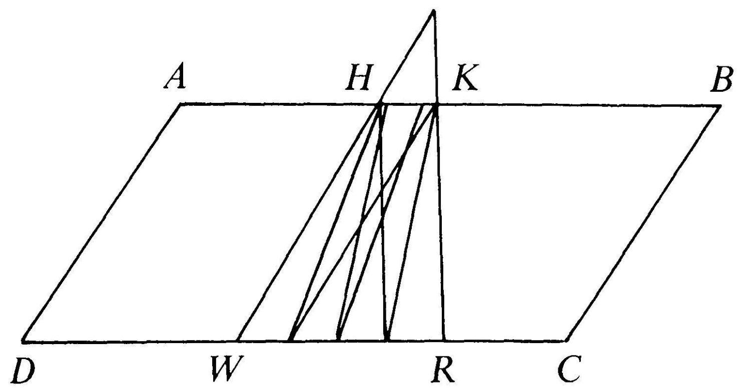

FIGURE 15

We wish to substitute the trapezoid $HKRW$ by strips, what we can easily do without augmenting the area in the way schematically indicated in Figure 15, where four parallelograms have sufficed to cover the portion of $ABCD$ covered by $HKRW$.

So we obtain ω_1, ω_2 and a number of small strips covering together an area smaller than $\varepsilon/2$. It is easy to see that for each segment joining A to a point of DC, there is another one parallel to it of the same length inside one of the strips we have obtained.

If we now perform an analogous construction starting from the other side BC of $ABCD$ we obtain a finite number of strips satisfying all the properties indicated (a), (b), (c).

The Besicovitch set is now very easily obtained as follows. We take a square $MNPQ$ of side length 1 and apply to it the above auxiliary construction (**) with $\varepsilon = 1/2$. We obtain a number of strips $\omega_1^1, \omega_2^1, \ldots, \omega_{r_1}^1$, covering an area smaller than 1/2 and such that for each segment determined by a point of MN and another of PQ there is a segment of the same length and direction inside $\Omega^1 = \omega_2^1 \cup \omega_2^1 \cup \ldots \cup \omega_{r_1}^1$.

Now we consider each of the parallelograms ω_j^1 and apply to it the same construction (**) with $\varepsilon = 1/2^2 r_1$. Collecting all parallelograms corresponding to each $\omega_j^1, j = 1, 2, \ldots, r_1$, we obtain a second family of parallelograms $\omega_1^2, \omega_2^2, \ldots, \omega_{r_2}^2$. Their union $\Omega^2 = \omega_1^2 \cup \omega_2^2 \cup \ldots \cup \omega_{r_2}^2$ has area less than 1/2, is contained in Ω^1 and, again, for each segment joining a point of MN to another of PQ there is another one of the same length and direction inside Ω^2. We proceed with the parallelograms ω_j^2 as we did with the ω_j^1, now with $\varepsilon = 1/2^3 r_2$, and so on. Thus we obtain

$$\Omega^1 \supset \Omega^2 \supset \Omega^3 \supset \ldots$$

of areas

$$S(\Omega^1) < 1/2, S(\Omega^2) < 1/2^2, S(\Omega^3) < 1/2^3, \ldots$$

The sets Ω^j are compact and have the property of containing a parallel translation of each segment with one extremity on MN and the other on PQ. The intersection

$$B = \Omega^1 \cap \Omega^2 \cap \ldots \cap \Omega^j \cap \ldots$$

is of null area and has this same property. We now proceed with the square $MNPQ$ in the same way in the other direction and obtain a compact set of measure zero containing a segment of length one in each direction, i.e. the Besicovitch set.

8. The Nikodym set

The Nikodym set can be obtained from the Perron tree in a similar way through the following auxiliary construction, also surprising in itself.