9. Mathematical frivolities? From the Perron tree to the measure of the density

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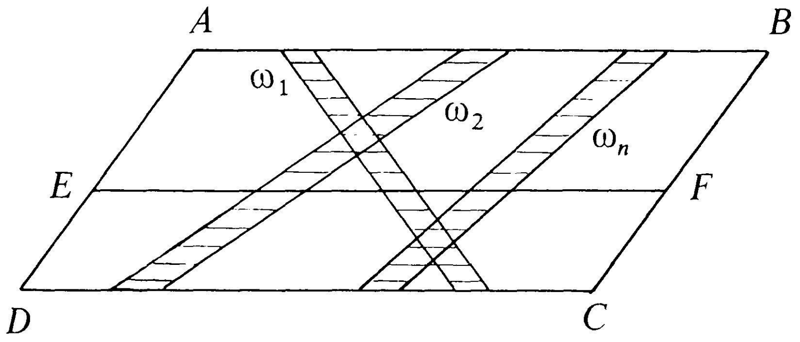

FIGURE 16

Let $ABCD$ *be an arbitrary parallelogram and* $CDEF$ *another one contained in it as Figure 16 shows. Let* ε *be any arbitrary positive number. Then one can construct a finite number of parallelograms* $\omega_1, \omega_2, \ldots, \omega_q$, *with a basis on* CD *and another one on* AB *such that the figure* $\omega_1 \cup \omega_2 \cup \ldots \cup \omega_q$ *covers* $CDEF$ *while the part of it above* EF *has area less than* ε, *that is:*

$$\omega_1 \cup \omega_2 \cup \ldots \cup \omega_q \supset CDEF$$
$$S((\omega_1 \cup \omega_2 \cup \ldots \cup \omega_q) \cap (ABCD - CDEF)) < \varepsilon$$

This construction is a little more technical than that of the Besicovitch set and will be omitted. For details we refer to Guzmán (1975).

9. MATHEMATICAL FRIVOLITIES? FROM THE PERRON TREE TO THE MEASURE OF THE DENSITY

What started as a puzzle has proved to have many important applications to solve some interesting problems of recent analysis.

Let us assume that we have a mass distributed on the plane and that we wish to measure the density of this distribution at each point. Let us also suppose that the mass is not continuously distributed. One can perhaps say: "Will it not be very artificial to consider a mass that is not continuously distributed?" It is true that the old Scholastic used to affirm that "*natura non facit saltus*" (nature does not proceed by jumps). However, the findings of modern physics permit us to affirm with even stronger motivation "*natura non facit nisi saltus*" (nature proceeds only by jumps). Therefore it is rather natural to consider a discontinuous mass distribution.

For a long time one thought that in order to measure the density one could take *any system of reasonable sets* that contract to the point at which one measures the density, find the mean density over such sets and hope that, when the sets become smaller and smaller, the mean density approaches a number, the density at the point. The Nikodym set shows in an easy way that one has to be very careful at choosing *reasonable sets.* As Zygmund observed (see the end of Nikodym's paper in 1927), it follows from the Nikodym set that if we take something apparently so reasonable as the system of all rectangles centered at the corresponding points, the mean densities can diverge. This, however, does not happen if the system is that of all circles or squares containing the points. Considerations of this type have given rise to the modern theory of differentiation of integrals.

10. Another fruit of the Perron tree. A problem on double Fourier series

A famous problem in Fourier analysis, open for a long time, has been recently solved in a rather simple way be the use of the Perron tree.

For a periodic function of two variables $f(x, y)$ of period 1 in each variable one can define its Fourier coefficients setting for $m = 0, \pm 1, \pm 2, \ldots, n = 0, \pm 1, \pm 2, \ldots$

$$a_{mn} = \int_0^1 \int_0^1 f(x, y) e^{-2\pi imx} e^{-2\pi iny} \, dx \, dy$$

and one can construct the corresponding Fourier series

$$\sum_{m,n} a_{mn} e^{2\pi imx} e^{2\pi iny}.$$

One can consider the partial sums of this series in several ways, in order to explore whether they converge or not to the original function. Thus, for example, one can consider the "square" sums

$$S_P f(x, y) = \sum_{\substack{|m| < P \\ |n| < P}} a_{mn} e^{2\pi imx} e^{2\pi iny}$$

or else the "rectangular" sums

$$S_{M,N} f(x, y) = \sum_{\substack{|m| < M \\ |n| < N}} a_{mn} e^{2\pi imx} e^{2\pi iny}$$

and examine whether in some sense $S_P f \to f$ as $P \to \infty$ or $S_{M,N} f \to f$ as $M, N \to \infty$. One can also consider the "circular" sums

$$S^R f(x, y) = \sum_{m^2 + n^2 \leqslant R^2} a_{mn} e^{2\pi imx} e^{2\pi iny}$$