1. Introduction

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **29 (1983)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **21.07.2024**

1. Introduction

Let κ be a partition of n, $\kappa = (\kappa_1, ..., \kappa_m)$, $\kappa_1 \geqslant \cdots \geqslant \kappa_m \geqslant 0$, $\Sigma\kappa_i = n$. We identify partitions $(\kappa_1, ..., \kappa_m)$ and $(\kappa_1, ..., \kappa_m, 0, ..., 0)$. Quite a few classes of objects in mathematics are of course classified by partitions and often inclusion, specialization or degeneration relations between these objects are described by a certain partial order on the set of partitions. This partial order on the set of all partitions of n is defined as follows:

$$(1.1)\qquad \begin{aligned}&(\kappa_1, ..., \kappa_m) > (\kappa_1', ..., \kappa_m')\\ &\text{iff}\quad \sum_{i=1}^{r} \kappa_i \leqslant \sum_{i=1}^{r} \kappa_i', \quad r = 1, ..., m.\end{aligned}$$

Thus, for example (2, 2, 1) > (3, 2). If $\kappa > \kappa'$ we say that κ specializes to κ' or that κ is more general than κ'. The reverse order has been variously called the dominance order [2], the Snapper order [34, 41] or the natural order [35]. It occurs naturally in several seemingly rather unrelated parts of mathematics. Some of these occurrences are the

(i) Snapper, Liebler-Vitale, Lam, Young theorem (on the permutation representations of the symmetric groups)

(ii) Gale-Ryser theorem (on existence of (0, 1)-matrices)

(iii) Muirhead's inequality (a symmetric mean inequality)

(iv) Gerstenhaber-Hesselink theorem (on orbit closure properties of SL_n acting on nilpotent matrices)

(v) Kronecker indices (on the orbit closure, or degeneration, properties of linear control systems acted on by the socalled feedback group)

(vi) Double stochastic matrices (when is a partition "an average" of another partition)

(vii) Shatz's theorem (on degeneration of vectorbundles over the Riemann sphere)

These will be described in more detail in section 2 below. In addition the same ordering, via the representation theory of the symmetric groups, plays a considerable role in theoretical chemistry (in the theory of chiral molecules, i.e., molecules that are optically active [10, 15, 17]. Finally the same order plays an important role in thermodynamical considerations. Consider an (isolated) system described by a probability vector $p = (p_1, p_2, \ldots)$, where p_i is the probability that a particle is in state i, evolving according to some "master equation". Then in [36, 37] it is shown that the system evolves in the direction of increasing $\bar{p} = (\bar{p}_1, \bar{p}_2, \ldots)$ (with respect to the specialization order), where $\bar{p}$ is the unique rearrangement of p such that $\bar{p}_1 \geqslant \bar{p}_2 \geqslant \ldots$. This statement is a good deal stronger, in fact infinitely stronger [38], than the statement that the entropy $-\sum_{i=1}^{\infty} p_i \ln p_i$ must always increase.

Certain occurrences of the specialization order are known to be intimately related. Thus (i), (ii), (iii) and (vi) are very much related [2, 5, 12], cf. also section 2 below, and so are (v) & (vii) [14] and section 8 below. This paper will show that all these manifestations of this order are intimately related. Their common meeting ground seems to be the ordering defined by closure relations of the Schubert-cells (with respect to a standard basis) of a Grassmann manifold. I.e. a Schubert-cell $SC(\lambda)$ is more general than $SC(\lambda')$; in symbols: $SC(\lambda) > SC(\lambda')$, iff $\overline{SC(\lambda)} \supset SC(\lambda')$. This order in turn is much related to the Bruhat ordering (sometimes called Bernstein-Gelfand-Gelfand ordering) on the Weyl group S_n. It is, in fact, the quotient ordering induced by the canonical map of the manifold of all flags in $\mathbf{R}^{n+m}$ to the Grassmann manifold of n-planes in $(n+m)$-space.

It should be said that in all probability there is much more to be said. The diagram of interrelations between the manifestations of the specialization order (cf. section 5.1 below) has overlap with another (functorial relationship) diagram centering around the irreducible quotients of Verma modules for sl_n, the Jantzen conjecture (now proved by A. Joseph) and the Bruhat ordering, and involving, among others, work of Kazhdan-Lusztig, Gelfand-MacPherson (relations with Schubert cells), Borho-Kraft and the same relation between orbits of nilpotent matrices and permutation representations which plays a role in this paper. (We owe these remarks to W. Borho).