

# 4. Schubert Cells

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the point of  $\mathbf{G}_n(\mathbf{C}^{n+m})$  represented by  $\text{Ker}(V \rightarrow E(x))$ . This gives a holomorphic map  $\Psi_E: M \rightarrow \mathbf{G}_n(\mathbf{C}^{n+m})$  such that the pullback of  $\xi_m$  by means of  $\Psi_E$  is isomorphic to  $E$ ,  $\Psi_E^! \xi_m \simeq E$ . It is universality properties such as this one which account for the importance of the bundles  $\xi_m$  and  $\eta_n$  in differential and algebraic topology [16], algebraic geometry and also system and control theory (cf. [22, 23] and the references therein for the last mentioned).

The bundle  $\xi_m$  has a number of obvious holomorphic sections, viz. the sections defined by  $\varepsilon_i(x) = e_i \bmod x$  where  $e_i$  is the  $i$ -th standard basis vector of  $\mathbf{C}^{n+m}$ ,  $i = 1, \dots, n+m$ . And, as a matter of fact, it is not difficult to show that  $\Gamma(\xi_m, \mathbf{G}_n(\mathbf{C}^{n+m}))$  is  $(n+m)$ -dimensional and that the  $\varepsilon_1, \dots, \varepsilon_{n+m}$  form a basis for this space of holomorphic sections; cf. subsection 8.1 below.

#### 4. SCHUBERT CELLS

4.1. *Schubert Cells.* Consider again the Grassmann manifold  $\mathbf{G}_n(\mathbf{C}^{m+n})$ . Let  $\underline{A} = (A_1, \dots, A_n)$  be a sequence of  $n$ -subspaces of  $\mathbf{C}^{m+n}$  such that  $0 \neq A_1 \subset A_2 \subset \dots \subset A_n$  with each containment strict. To each such sequence  $\underline{A}$  we associate the closed subset

$$(4.2) \quad SC(\underline{A}) = \{x \in \mathbf{G}_n(\mathbf{C}^{m+n}) \mid \dim(x \cap A_i) \geq i\}$$

and call it the closed Schubert-cell of the sequence  $\underline{A}$ . In particular if

$$0 < \gamma_1 < \gamma_2 < \dots < \gamma_n \leq n + m$$

is a strictly increasing sequence of natural numbers less than or equal to  $n + m$  then we define (setting  $\gamma = (\gamma_1, \dots, \gamma_n)$ )

$$(4.3) \quad SC(\gamma) = SC(\mathbf{C}^{\gamma_1}, \dots, \mathbf{C}^{\gamma_n})$$

where  $\mathbf{C}^r$  is viewed as the subspace of all vectors in  $\mathbf{C}^{n+m}$  whose last  $n + m - r$  coordinates are zero.

4.4 *Flag Manifolds and the Bruhat Decomposition.* A flag in  $\mathbf{C}^{n+m}$  is a sequence of subspaces  $\underline{F} = F_1 \subset \dots \subset F_{n+m} \subset \mathbf{C}^{n+m}$  such that  $\dim F_i = i$ . Let  $Fl(\mathbf{C}^{n+m})$  denote the analytic manifold of all flags in  $\mathbf{C}^{n+m}$ . There is a natural holomorphic mapping  $Fl(\mathbf{C}^{n+m}) \rightarrow \mathbf{G}_n(\mathbf{C}^{n+m})$  given by associating to a flag  $\underline{F}$  its  $n$ -th element  $F_n$ . The flag manifold can be seen as the space of all cosets  $Bg$ ,  $g \in \mathbf{GL}_{n+m}(\mathbf{C})$  where  $B$  is the Borel subgroup of all lower triangular matrices

in  $GL_{n+m}(\mathbb{C})$ . The mapping  $GL_{n+m}(\mathbb{C}) \rightarrow Fl(\mathbb{C}^{n+m})$  associates to a matrix  $g$  the flag  $\underline{F}(g)$  whose  $i$ -th element is the subspace of  $\mathbb{C}^{n+m}$  spanned by the first  $i$  row vectors of  $g$ .

Now view  $S_{n+m}$ , the symmetric group on  $n + m$  letters as a subgroup of  $GL_{n+m}(\mathbb{C})$  by letting it permute the basis vectors ( $\sigma(e_i) = e_{\sigma(i)}$ ). Then in  $GL_{n+m}(\mathbb{C})$  we have the so-called Bruhat decomposition.

$$(4.5) \quad GL_{n+m}(\mathbb{C}) = \bigcup_{\sigma} B \sigma B \quad (\text{disjoint union})$$

Where  $\sigma$  runs through the Weyl group  $S_{n+m}$  of  $GL_{n+m}(\mathbb{C})$ . An analogous decomposition holds in a considerably more general setting (reductive groups, cf. [24], section 28).

4.6. *The Bruhat order (also sometimes called Bernstein-Gelfand-Gelfand, or BGG order).* The closure of a double coset  $B \sigma B$  is necessarily a union of other double cosets (by continuity). This defines an ordering on the Weyl group  $S_{n+m}$  defined by

$$(4.7) \quad \sigma > \tau \leftrightarrow \overline{B \sigma B} \supset B \tau B$$

This ordering plays a considerable role in the study of cohomology of flag spaces [1] and also in the theory of highest weight representations [25, 26].

Let  $H$  be the subgroup of  $G_{n+m}(\mathbb{C})$  consisting of all block lower triangular matrices of the form  $\begin{pmatrix} S_{11} & 0 \\ S_{21} & S_{22} \end{pmatrix}$ ,  $S_{11} \in GL_n(\mathbb{C})$ ,  $S_{22} \in GL_m(\mathbb{C})$ ,  $S_{21}$  an arbitrary  $m \times n$  matrix. Then, using the remarks made in subsection 4.4 above, one sees that  $G_n(\mathbb{C}^{n+m})$  is the coset space  $\{Hg \mid g \in GL_{n+m}(\mathbb{C})\}$ . Now let  $\sigma \in S_{n+m}$  and let  $\gamma_1 < \dots < \gamma_n$  be the  $n$  natural numbers in increasing order determined by

$$\sigma(e_{\gamma_i}) \in \{e_1, \dots, e_n\}, i = 1, \dots, n.$$

Then one easily sees that the image of  $B \sigma B$  under  $GL_{n+m}(\mathbb{C}) \rightarrow G_n(\mathbb{C}^{n+m})$ , i.e. the set of all spaces spanned by matrices of the form  $h \sigma b$ ,  $h \in H$ ,  $b \in B$ , is the open Schubert cell of all elements in  $G_n(\mathbb{C}^{n+m})$  spanned by the rows of a matrix of the form

$$\begin{array}{cccccc} * & \dots & * & 0 & \dots & 0 & 0 & \dots & 0 \\ * & \dots & * & * & \dots & * & 0 & \dots & 0 \\ * & \dots & * & * & \dots & * & \dots & * & 0 \dots 0 \end{array}$$

where the last \* in each row is nonzero. The closure of this open Schubert-cell is the Schubert-cell  $SC(\gamma)$  defined in (4.3) above.

One easily checks that

$$(4.8) \quad SC(\mu) \subset SC(\gamma) \leftrightarrow \mu_i \leq \gamma_i, i = 1, \dots, n$$

and this order on the Schubert cells  $SC(\gamma)$ , or the equivalent ordering on  $n$ -tuples of natural numbers, is therefore a quotient of the Bruhat order on the Weyl group  $S_{n+m}$ . It is the induced order on the set of cosets  $(S_n \times S_m)\sigma$ ,  $\sigma \in S_{n+m}$ . (Obviously if  $\tau \in S_n \times S_m$ , then  $\tau\sigma(e_{\gamma_i}) \in \{e_1, \dots, e_n\}$  if  $\sigma(e_{\gamma_i}) \in \{e_1, \dots, e_n\}$ .) (And inversely the Bruhat order is determined by the associated orders of Schubert cells in the sense that  $\sigma > \tau$  in  $S_n$  iff for all  $k = 1, \dots, n - 1$  we have for the associated Schubert cells in  $G_k(\mathbb{C}^n)$  that  $SC(\sigma) \supset SC(\tau)$ ; this is a rather efficient way of calculating the Bruhat order on the Weyl group  $S_n$ .)

## 5. INTERRELATIONS

Now that we have defined the concepts we need we can start to describe some interrelations between the various manifestations of the specialization order we discussed in section 2 above.

5.1. *Overview of the Various Relations.* A schematic overview of the various interconnections is given by the following diagram. In this diagram we

