11. A FAMILY OF REPRESENTATIONS OF S_{n+m} PARAMETRIZED BY $G_n(C^{n+m})$

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10.3. *The Inverse Result.* Inversely if ρ_0 is a subrepresentation of ρ then there is a family of representations (10.3) such that $Im\,\pi_t \simeq \rho$ for $t \neq 0$ and $Im\,\pi_0 \simeq \rho_0$, and if ρ is generated (as a $\mathbf{C}[G]$-module) by one element one can take for M in (10.2) the regular representation. Indeed if ρ_0 is a subrepresentation of ρ then $\rho = \rho_0 \oplus \rho_1$. Let $\pi : M \to \rho = \rho_0 \oplus \rho_1$ be a surjective map of representations. Let π_0, π_1 be the two components of π. Let $s = (s_0, s_1)$ be a section of π. Then $\pi_0 s_0 = id, \pi_1 s_1 = id, \pi_0 s_1 = 0, \pi_1 s_0 = 0$ and it follows that $\pi(t)$ consisting of the components π_0 and $t\pi_1$ is still surjective. Hence $Im\,\pi(t) = \rho$ and $Im\,\pi(0) = \rho_0$.

11. A FAMILY OF REPRESENTATIONS OF S_{n+m} PARAMETRIZED BY $\mathbf{G}_n(\mathbf{C}^{n+m})$

11.1. *Construction of the Family.* Let M be the regular representation of S_{n+m}. That is M has a basis e_σ, $\sigma \in S_{n+m}$ and S_{n+m} acts on M by the formula $\tau(e_\sigma) = e_{\tau\sigma}$, for all $\tau \in S_{n+m}$. Now consider the universal bundle ξ_m over $\mathbf{G}(\mathbf{C}^{n+m})$ and the $n + m$ holomorphic section $\varepsilon_1, ..., \varepsilon_{n+m}$ defined by

$$\varepsilon_i(x) = e_i \bmod x \in \mathbf{C}^{n+m}/x\,,$$

where e_i is the i-th standard basis vector. Take the $(m+n)$-fold tensor product of ξ_m and define a family of homomorphisms parametrized by $\mathbf{G}_n(\mathbf{C}^{n+m})$ by

$$\pi_x : M \to \xi_m(x)^{\otimes(n+m)},\ e_{\sigma^{-1}} \mapsto \varepsilon_{\sigma(1)}(x) \otimes ... \otimes \varepsilon_{\sigma(n)}(x) \tag{11.2}$$

More precisely (11.2) defines a homomorphism of vectorbundles

$$\mathbf{G}_n(\mathbf{C}^{n+m}) \times M \to \xi_m^{\otimes(n+m)} \tag{11.3}$$

The group S_{n+m} acts on $\xi_m(x)^{\otimes(n+m)}$ by permuting the factors and it is a routine exercise to see that π_x is equivariant with respect to this action, i.e. that $\pi_x(\tau v) = \tau\pi_x(v)$ for all $v \in M, \tau \in S_{n+m}$. (Here the product $\tau\sigma \in S_{n+m}$ is interpreted as first the automorphism σ of $1, ..., n + m$ and then the automorphism τ.)

Thus $Im\,\pi_x = \pi(x)$ is a representation of S_{n+m} for all x giving us a family of representations parametrized by $\mathbf{G}_n(\mathbf{C}^{n+m})$. Fixing a point $x_0 \in \mathbf{G}_n(\mathbf{C}^{n+m})$ and choosing m independent sections of ξ_m in a neighbourhood U of x_0, this gives us families of homomorphisms of representations

(11.4) $$M \overset{\pi_x}{\rightarrow} (\mathbf{C}^m)^{\otimes(n+m)},\ x \in U \subset \mathbf{G}_n(\mathbf{C}^{n+m})$$

such that $Im\ \pi'_x \simeq \pi(x)$ for $x \in U$.

11.5. *Permutation Representations and Schubert-cells. (On connection D.)* Let $x \in \mathbf{G}_n(\mathbf{C}^{n+m})$ be a subspace of C^{n+m} spanned by the rows of a matrix of the form ($n = 3$, $n = 5$)

$$\begin{matrix} * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & * & * & 0 \end{matrix}$$

where all the *'s are nonzero. Then obviously the representation $\pi(x)$ of S^* is isomorphic to $\rho(\tilde{\kappa})$ with $\tilde{\kappa} = (4, 3, 1)$. Note that x is in the standard Schubert-cell $SC(\tau(\kappa))$, with $\kappa = (3, 2, 0)$. This holds in general and it is not difficult to extend this to

11.6. *Proposition.* Let κ be an m-part partition of n, $\tilde{\kappa} = (\kappa_1 + 1, ..., \kappa_m + 1)$. Then for almost all $x \in SC(\tau(\kappa))$, the representation of $\pi(x)$ of S_{n+m} contains the representation $\rho(\tilde{\kappa})$ and for some $x \in SC(\tau(\kappa))$, $\pi(x) \simeq \rho(\tilde{\kappa})$.

Conjecturally the reverse holds also. That is if for almost all x in a standard Schubert-cell $SC(\lambda)$ we have that $\pi(x)$ contains $\rho(\tilde{\kappa})$ then $\lambda_i \geqslant \tau_i(\kappa)$, $i = 1, ..., n$. And I am even inclined to believe that if $x \in SC(\lambda)$ and $\pi(x)$ contains (or is equal to) $\rho(\tilde{\kappa})$ then $\lambda_i \geqslant \tau_i(\kappa)$.

Note also that the matrices (11.5) are precisely the type of matrices $(sI \dot{-} A; B)$ for a system $\Sigma = (A, B)$ in feedback canonical form ($s \neq 0, \infty$) suggesting that there is a natural representation of S_{n+m} attached to Σ awaiting interpretation.

11.7. *On a proof of the Snapper, Liebler-Vitale, Lam, Young Theorem via the Universal Family* (11.2). The structure of the family of representations (11.2) rather quickly suggests a way of proving the Snapper etc. theorem by deformation arguments as in 10.1. The argument is, however, more complicated than one would like perhaps. It is perhaps best illustrated by means of an example.

Consider an $x \in \mathbf{G}_3(\mathbf{C}^5)$ spanned by the rows of a matrix of the form

$$\begin{matrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ z & 0 & 0 & -1 & t \end{matrix}$$

Let $f_1, \ldots, f_5$ be the images of the standard basis vectors $e_1, \ldots, e_5$ in $\mathbf{C}^5/x$. Then $f_1 = f_2 = f_3 \neq f_4 = zf_1 + tf_5$ so that f_1 and f_5 are a basis for $\mathbf{C}^5/x$ for all values of z and t. Let $(1) \in S_5$ be the identity permutation. The image of $e_{(1)} \in M$ in $(\mathbf{C}^5/x)^{\otimes 5}$ is by the definition (11.2) equal to

(11.8) $$f_1 \otimes f_2 \otimes f_3 \otimes f_4 \otimes f_5 = zf_{11115} + tf_{11155}$$

Where f_{11115} is short for $f_1 \otimes f_1 \otimes f_1 \otimes f_1 \otimes f_5$ and similarly for other 5-tuples of indices. Symmetrizing the element (11.8) with respect to the permutation (45) gives us

(11.9) $$z(f_{11115}+f_{11151}) + 2tf_{11155}$$

Let V_1 be the subrepresentation of $Im\, \pi_x$ generated by the element (11.9). (The representation $Im\, \pi_x$ is the subrepresentation of $(\mathbf{C}^5/x)^{\otimes 5}$ generated by (11.8).) Now (11.9) is invariant under the Young subgroup $S_3 \times S_2$. Hence $\dim V_1 \leqslant 5!/3!2!$. On the other hand, if $t \neq 0$ then setting $z = 0$ in (11.9) (which corresponds to the surjective map mentioned just above 10.2 associated to a family of representations) obviously maps V_1 onto the vector space with as basis all symbols $f_{\ldots}$ with three of the indices equal to 1 and 2 equal to 5. This is $\rho(3, 2)$ of dimension $5!/3!2!$ so that $V_1 \simeq \rho(3, 2)$ if $t \neq 0$. Now for $z \neq 0$ set $t = 0$ in (11.8) to obtain a homomorphism of representations

$$Im\, \pi_x \to \pi(4, 1)$$

It is now not hard to prove that (cf. [6] for a detailed proof).

11.10. *Proposition.* The composed homomorphism of representations

$$\rho(3, 2) \simeq V_1 \subset Im\, \pi_x \to \rho(4, 1)$$

is surjective.

This then proves that $\rho(4, 1)$ is a direct summand of $\rho(3, 2)$. The argument generalizes without difficulty for partitions $\kappa > \lambda$ such that λ is obtained from κ by a transformation of the type described in 6.7 above.

This is by no means the easiest way to prove this theorem. It is perfectly easy to describe the morphism $\rho(\kappa) \to \rho(\lambda)$ directly and then the general analogue of proposition 11.10 yields the Snapper, Liebler-Vitale, Lam, Young result. This proof uses no representation theory at all (except the definition of the permutation representations $\rho(\kappa)$; cf. [6] for details).

11.11. *Remarks.* It is conceivable that the family (11.2) contains all the families of representations one needs to prove the Snapper etc. result by means of deformation arguments. Quite generally we would like to pose the question which representations occur in this family and investigate universal families (for continuous families) of homomorphisms of representations from some fixed representation space into another.

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