Part II: Statement of the theorem

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When this estimate is substituted into the Erdös-Turan inequality with $m = [p^{\frac{1}{4}}]$, we get

$$\operatorname*{Sup}_J |(p-2)^{-1}A(p, J) - |J|| \leqslant \frac{1}{m+1} + \frac{8}{\pi} mp^{-\frac{1}{2}} \ll p^{-\frac{1}{4}}.$$

This establishes the result. A comparison of the estimate $A(p, J) = p|J| + O(p^{\frac{3}{4}})$ with some of the classical prime number theorems suggests that perhaps the stronger result

$$A(p, J) = p|J| + O(p^{\frac{1}{2}+\varepsilon})$$

should be true.

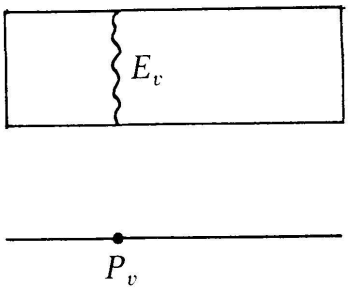
PART II: STATEMENT OF THE THEOREM

§1.1. INTRODUCTION. In the statement of Deligne's theorem there appear certain Euler products which are generalizations of the Artin-Grothendieck L-functions and which satisfy some rather natural growth conditions; these conditions are stated below in §2 as Axioms A and B. In order to elucidate the applicability of the theorem, to introduce some relevant concepts from representation theory, and to prepare the notation that goes into the statement of the theorem, we now give two examples one of a geometric nature, the other of an arithmetic nature. The expert will realize that both examples are intimately connected, say via the Selberg-trace Formula.

§1.2. GEOMETRIC EXAMPLE. As in Part I, let $\mathbf{F}_q$ be the finite field of q elements and let $A = \mathbf{F}_q[T]$ be the coordinate ring of the affine line $\mathbf{A}^1$. For technical reasons and to simplify our presentation, we assume the characteristic of $\mathbf{F}_q$ is not 2 or 3. The closed points on the affine line $\mathbf{A}^1$ are in one-to-one correspondence with the irreducible monic polynomials in A. Now if $P = P_v$ is such an irreducible polynomial in A, then the image of T under the reduction map

$$\begin{aligned} A &\to A/(P) = \mathbf{F}_{q_v} \\ T &\to t_v, \end{aligned}$$

gives an element t_v in the finite field $\mathbf{F}_{q_v}$ with $q_v = q^{\deg(P)}$ elements. We can now consider the elliptic family



$$\begin{array}{l} E : y^2 = x(x-1)(x-T)\,, \\ \downarrow \\ \mathbf{A}^1 \end{array}$$

where $E_v : y^2 = x(x-1)(x-t_v)$ is the fiber in E above the point P_v. If we exclude from $\mathbf{A}^1$ the points corresponding to the polynomials $P_r = T$, $P_v = T - 1$, then each fiber E_v is an elliptic curve defined over the finite field $\mathbf{F}_{q_v}$. A well known theorem of Hasse established in 1934 states that

$$\#\{(x, y) \in (\mathbf{F}_{q_v})^2 \mid y^2 = x(x-1)(x-t_v)\} = q_v - (\alpha_v + \beta_v) + 1\,,$$

where

$$\alpha_v = q_v^{\frac{1}{2}} e^{i\theta_v}\,, \qquad \beta_v = q_v^{\frac{1}{2}} e^{-i\theta_v}\,,$$

where $\theta_v \in [0, 2\pi)$.

Let $SU(2)$ be the group of special unitary matrices of size 2×2 and consider the trivial extension

$$0 \to SU(2) \to G \to \mathbf{Z} \to 0$$

given by the direct product $G = SU(2) \times \mathbf{Z}$. Let Σ be the set of all irreducible monic polynomials in $A = \mathbf{F}_q[T]$. For each $v \in \Sigma$ we have an element in G

$$\left\{\begin{pmatrix} e^{i\theta_v} & 0 \\ 0 & e^{-i\theta_v} \end{pmatrix}, -\deg v\right\};$$

denote by x_v the conjugacy class of this element in $G = SU(2) \times \mathbf{Z}$. Let ω_1 be the quasi-character

$$\omega_1 : \mathbf{Z} \to \mathbf{R}_+$$

which sends the integer n to $\omega_1(n) = q^n$, and for s a complex number put $\omega_s = \omega_1^s : \mathbf{Z} \to \mathbf{C}^*$; this gives by composition with the projection map $G \to \mathbf{Z}$ a representation

$$\omega_s : G \to \mathbf{C}^*\,.$$

The finite dimensional representations of $SU(2)$ are well known; they have the following structure: for each positive integer k, there is a representation

$$\mathrm{Sym}^k\, r : SU(2) \to GL(k+1, \mathbf{C})\,.$$

For $k = 0$, this is the trivial representation of $SU(2)$; for $k = 1$ $\mathrm{sym}^1\, r = r$ is just the standard representation which sends an element in $SU(2)$ into the same element in $GL(2, \mathbf{C})$. In general, if $g = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \in SU(2)$ then $\mathrm{sym}^k\, r(g)$ is the diagonal matrix in $GL(k+1, \mathbf{C})$ given by

$$\mathrm{Sym}^k\, r(g) = \mathrm{Diag}\,[\alpha^k, \alpha^{k-1}\beta, \ldots, \alpha\beta^{k-1}, \beta^k]\,.$$

It can easily be shown that the set of all finite dimensional representations of the locally compact group G are of the form

$$\tau = (\mathrm{Sym}^k r)\cdot \omega_s\,,$$

for some positive integer k and a complex number s; for such a representation, if $s = \sigma + it$, we call σ the real part of τ and write

$$R(\tau) = \sigma\,.$$

In particular if τ is an arbitrary representation then $R(\tau\cdot\omega_s) = R(\tau) + R(s)$. With the above notations we now associate to each representation τ of G the L-function

$$L(\tau) = \prod_{v\in\Sigma} \frac{1}{\det(I-\tau(x_v))}\,;$$

an easy comparison of $L(\tau)$ with the zeta function $Z(s, \mathbf{A}^1)$ of §1 of Part I shows that $L(\tau)$ converges absolutely if $R(\tau) > 1$. It is a consequence of Grothendieck's Trace formula that $L(\tau)$ has a holomorphic continuation to the region $R(\tau) \geqslant 1$ except for a simple pole at $\tau = \omega_1$. Deligne's generalization of the method of Hadamard and de la Vallée-Poussin will imply that

$$L(\tau) \neq 0 \quad \text{for all } \tau \text{ with} \quad R(\tau) = 1\,.$$

From here on one takes the familiar road of analytic number theory and applies criteria of the Weyl-type as well as Tauberian theorems to obtain equidistribution results. ([9], [12].)

§1.3. ARITHMETIC EXAMPLE. Let us consider our favorite arithmetic function: the Ramanujan function $\tau(n)$ which is defined by the formal expansion

$$x \prod_{n=1}^{\infty} (1-x^n)^{24} = \sum_{n=1}^{\infty} \tau(n)x^n\,.$$

Let Σ denote the set of rational primes. For each prime $p \in \Sigma$ it follows from Deligne's proof of the Ramanujan conjecture that

$$\tau(p) = \left(e^{i\theta_p} + e^{-i\theta_p}\right)p^{11/2}\,,$$

with $\theta_p \in [0, 2\pi)$. In this arithmetic situation we consider the trivial group extension

$$0 \to SU(2) \to G \to \mathbf{R} \to 0$$

given by the direct product $G = SU(2) \times \mathbf{R}$. With each prime p we associate the element

$$\left\{ \begin{pmatrix} e^{i\theta_p} & 0 \\ 0 & e^{-i\theta_p} \end{pmatrix}, -\log p \right\},$$

and denote by x_p the conjugacy class in G which contains it. Let ω_1 be the quasi-character

$$\omega_1 : \mathbf{R} \to \mathbf{R}_+^*$$
$$r \to \omega_1(r) = e^r ;$$

for each complex number s, let ω_s be the 1-dimensional complex representation

$$\omega_s : G \to \mathbf{C}^*$$

obtained by composing ω_1^s with the projection map $G \to \mathbf{R}$. Again it is not very difficult to show that all the finite dimensional representations of G are of the form

$$\tau = (\mathrm{sym}^k r) \cdot \omega_s$$

for some positive integer k and a complex number s. For such a representation τ with $s = \sigma + it$, we put $R(\tau) = \sigma$ and call it the real part; it is clear that we have $R(\tau \cdot \omega_s) = R(\tau) + R(s)$. With the above notation, and with τ a finite dimensional representation of G, we define an L-function

$$L(\tau) = \prod_{p \in \Sigma} \frac{1}{\det(I - \tau(x_p))};$$

a comparison of this L-function with the ordinary Riemann zeta function shows that it is absolutely convergent for $R(\tau) > 1$. It is known that $L(\tau)$ has a holomorphic continuation to the region $R(\tau) \geqslant 1$ for $\tau = (\mathrm{sym}^k r) \cdot \omega_s$ with $k = 1, 2, 3$ and possibly other values not known to the author. Clearly $L(\omega_s) = \zeta(s)$ and so it has a simple pole at $s = 1$. If it could be established that $L(\tau)$ has a holomorphic continuation to $R(\tau) \geqslant 1$ for all representations $\tau = (\mathrm{sym}^k r) \cdot \omega_s$, $k \geqslant 1$, then Deligne's generalization of the method of Hadamard and de la Vallée-Poussin would imply that

$$L(\tau) \neq 0 \quad \text{for all } \tau \text{ with } \quad R(\tau) = 1 .$$

By well known techniques in analytic number theory [9], it would then be possible to prove

The Sato-Tate Conjecture: for large x

$$\sum_{p \leq x} \chi(\theta_p) \sim \frac{2}{\pi} \int_J (\sin \theta)^2 d\theta \cdot \frac{x}{\log x},$$

where χ is the characteristic function of the subinterval $J \subset [0, 2\pi)$.

§2. THE GENERAL SETTING: AXIOMS A AND B. Deligne's generalization of the Hadamard and de la Vallée-Poussin method applies to a broad class of L-functions which are subjected to two basic axioms. Before we give the statement of the main result we introduce some notation and define the class of L-functions that will be considered.

Let Γ be a group which is isomorphic to $\mathbf{Z}$ or to $\mathbf{R}$. Let $\omega_1 : \Gamma \to \mathbf{R}_+^*$ be a non-trivial quasi-character. Let G be a locally compact group which is an extension of Γ by a compact group $\overset{\circ}{G}$:

$$0 \to \overset{\circ}{G} \to G \to \Gamma \to 0 .$$

Σ will denote an infinite countable set, and $(x_v)_{v \in \Sigma}$ will be a family of conjugacy classes in G indexed by Σ. The examples of the previous section motivate the following restrictions on the above data.

Axiom A (i) If Γ is isomorphic to $\mathbf{R}$, the extension G is trivial.

(ii) If Γ is isomorphic to $\mathbf{Z}$, the center of G is mapped onto a subgroup of finite index in $\mathbf{Z}$.

It should be observed that since $H^2(\mathbf{R}, \overset{\circ}{G}) = \{1\}$ for any compact group $\overset{\circ}{G}$, the condition $A(i)$ is automatically satisfied, i.e. $G = \overset{\circ}{G} \times \mathbf{R}$ a direct product. One of the many applications that Deligne makes of his main result is to the proof of the Weil conjecture. In this situation it suffices to consider the case where G is the direct product of $\Gamma = \mathbf{Z}$ by a compact Lie group $\overset{\circ}{G}$, whose connected component of the identity $\overset{\circ\circ}{G}$ is semisimple.

The condition $A(ii)$ is not really necessary in the proof of the main result; what does seem to be needed is some sort of control on the growth of the matrix coefficients $\rho_{ij}(g)$ of a continuous finite dimensional representation $\rho : G \to GL(V_{\mathbf{C}})$, for example the boundedness of the matrix coefficients $\rho_{ij}(g)$ will guarantee that the representation ρ is unitarizable. Below we shall see that actually polynomial growth as measured by a power of $\omega_1(g)$ will suffice. In the proof of the Weil conjecture the group G admits a linear representation whose restriction to $\overset{\circ}{G}$ has a finite kernel; for this type of group G it can be shown that A(ii) is automatically satisfied.

With the non-trivial quasi-character $\omega_1 : \Gamma \to \mathbf{R}_+^*$, we associate a family of morphisms

$$\omega_s : G \xrightarrow{pr} \Gamma \to \mathbf{C}^* ,$$

parametrized by complex numbers $s \in \mathbf{C}$:

$$\omega_s(g) = \omega_1(pr(g))^s .$$

The norm of an element $v \in \Sigma$ is defined by $N_v = \omega_{-1}(x_v)$. If Γ is isomorphic to $\mathbf{Z}$, then $\{\omega_1(\gamma) : \gamma \in \Gamma\}$ is a discrete cyclic subgroup of $\mathbf{R}_+^*$ and hence of the form $\{q^{\mathbf{Z}}\}$, where q is a positive real number > 1. This gives rise to an isomorphism

$$\deg : \Gamma \to \mathbf{Z}$$

whose sign we select so that $\omega_1(\gamma) = q^{-\deg(\gamma)}$. We also denote by deg the morphism

$$\deg : G \to \Gamma \to \mathbf{Z}$$

Obtained by composing the projection map $G \to \Gamma$ with deg. In the following we define the degree of an element $v \in \Sigma$ by $\deg(v) = \deg(x_v)$.

In case $\Gamma \simeq \mathbf{Z}$, Axióm A implies there is an element g in the center of G whose image in Γ is non-trivial. Weyl's unitary trick can be used to show that a complex linear representation $\tau : G \to GL(V)$ is equivalent to a unitary representation if and only if $\tau(g)$ is. In fact if ψ is a Hermitian structure on V which is invariant under g, i.e.

$$\psi(\tau(g) \cdot v, \tau(g) \cdot w) = \psi(v, w) , \qquad v, w \in V,$$

then integration over the compact group $H = G/g^{\mathbf{Z}}$ gives a G-invariant form

$$\tilde{\psi}(v, w) = \int_H \psi(\tau(g) \cdot v, \tau(g) \cdot w) dg ,$$

which also defines a Hermitian structure on V. Hence τ is equivalent to a unitary representation.

Consider now the general situation. Let $\tau : G \to GL(V)$ be an irreducible complex linear representation. Let ψ define a Hermitian structure on V. If g belongs to the center, then Schur's Lemma implies $\tau(g)$ is a scalar multiple of the identity. Hence there is a complex number λ such that

$$\psi(\tau(g) \cdot v, \tau(g) \cdot w) = |\lambda|^2 \psi(v, w) .$$

Denote by σ the real number such that $|\lambda| = \omega_1(g)^\sigma = \omega_\sigma(g)$ and observe that the Hermitian form

$$\psi(\tau \cdot \omega_{-\sigma}(g) \cdot v, \tau \cdot \omega_{-\sigma}(g) \cdot w)$$

is now invariant under the action of the center of G. Integration over the quotient of G by its center gives a G-invariant Hermitian form. Therefore the representation $\tau\omega_{-\sigma}$ is equivalent to a unitary representation. The number σ will be called the real part of the representation τ and is denoted $R(\tau)$. If τ is unitary, then $R(\tau) = 0$ and also $R(\tau\omega_s) = R(\tau) + R(s)$.

The irreducible representations of G of the form $\tau \cdot \omega_s$ with τ unitary will be called quasi-unitary. We denote by $\tilde{G}$ the family of isomorphism classes of irreducible quasi-unitary representations of G; we let $\hat{G}$ be the subfamily of those which are unitary. On $\tilde{G}$ we consider the equivalence relation: $\tau, \tau' \in \tilde{G}$ are equivalent if τ is in the class of $\tau' \cdot \omega_s$ for some $s \in \mathbf{C}$. Under this equivalence relation $\tilde{G}$ is partitioned into a disjoint union

$$\tilde{G} = \bigcup_{\tau \in \hat{G}} \{\tau \cdot \omega_s \mid s \in \mathbf{C}\} .$$

By introducing the parameter s, we may now view an equivalence class of quasi-unitary representations as a Riemann surface. In fact the map $s \to \tau \cdot \omega_s$ identifies the set $\{\tau \cdot \omega_s \mid s \in \mathbf{C}\}$ with

i) The complex plane $\mathbf{C}$ if $\Gamma \simeq \mathbf{R}$ or

ii) with the strip $\mathbf{C} / \dfrac{2\pi i}{\log q} \mathbf{Z}$, if $\Gamma \simeq \mathbf{Z}$ and q is the real number with $\omega_1(\gamma) = q^{-\deg \gamma}$.

As is well known, by viewing $\tilde{G}$ as a collection of Riemann surfaces, it makes sense to talk about the regularity of a function of quasi-unitary representations at a point or in a region, or about its singularities. The question of analytic continuation, when considered on each connected surface, also makes sense.

Remark. It is in the above spirit that the zeros of an L-function should be considered as a discrete set of quasi-unitary representations on the same connected component, and the explicit formulas of number theory should be considered as generalized trace formulas.

Axiom B (i) For every $v \in \Sigma$, one has $Nv > 1$.

(ii) The infinite product $\prod_{v \in \Sigma} (1 - Nv^{-s})^{-1}$ converges absolutely for $R(s) > 1$.

For Γ isomorphic to $\mathbf{Z}$, the first relation means: $\deg(v) > 0$; B(ii) means that

$$\sum_{m=1}^{\infty} \frac{1}{m} \left\{ \sum_{d \mid m} dN_d \right\} q^{-ms} ,$$

where

$$N_m = \# \{v \in \Sigma \mid \deg(v) = m\} ,$$

which is the logarithm of the infinite product, converges absolutely for $R(s) > 1$, that is to say for every $\varepsilon > 0$

$$N_m = O(q^{(1+\varepsilon)m}).$$

The condition B(ii) assures that for every $\tau \in \tilde{G}$, the infinite product

$$L(\tau) = \prod_{v \in \Sigma} \frac{1}{\det(I - \tau(x_v))}$$

converges absolutely for $R(\tau) > 1$. Also each factor is holomorphic in τ for $R(\tau) > 0$, and the function $L(\tau)$ is holomorphic for $R(\tau) > 1$ and does not vanish in this region. In the following we put $L(s, \tau) = L(\tau \cdot \omega_s)$.

§3. THEOREM (Deligne). *With the assumptions and notations as above, suppose that* $L(\tau)$ *as a function of* τ *has a meromorphic continuation to* $R(\tau) \geqslant 1$, *and that in this region* $R(\tau) \geqslant 1$ *it is holomorphic except for a simple pole at* ω_1. *Then the function* $L(\tau)$ *does not vanish for* $R(\tau) = 1$, *except possibly for at most one representation* τ_0, *of dimension* 1 *and defined by a character* $\omega_1\varepsilon$ *with* ε *of order* 2.

§4. THE MAIN LEMMA. For a complex linear representation $\tau : G \to GL(V)$, of dimension d, not necessarily irreducible, we have associated the zeta function

$$L(\tau) = \prod_v L_v(\tau),$$

where

$$L_v(\tau) = \frac{1}{\det(I - \tau(x_v))} = \prod_{i=1}^{d} \frac{1}{1 - \beta_i(v)},$$

and $\beta_1(v), \ldots, \beta_d(v)$ are the eigenvalues of a matrix in the conjugacy class of $\tau(x_v)$. Now for s a complex number we put

$$L(\tau, s) = L(\tau \cdot \omega_s)$$

and define

$$L'(\tau) = \frac{d}{ds} L(\tau\omega_s)\,|_{s=0}.$$

In particular, in the domain of absolute convergence for the product

$$L(\tau\omega_s) = \prod_{v \in \Sigma} \prod_{i=1}^{d} \frac{1}{1 - \beta_i(v) N v^{-s}},$$

that is to say for $R(\tau\omega_s) > 1$, we can take the logarithmic derivative with respect to the complex variable s and obtain

$$-\frac{L'}{L}(\tau\omega_s) = \sum_{\substack{v\in\Sigma \\ n>0}} (\log Nv) \cdot Nv^{-sn}\, \chi_\tau(x_v^n)\,.$$

If we let $s = 0$ in the above formula, we obtain for $R(\tau) > 1$

$$-\frac{L'}{L}(\tau) = \sum_{\substack{v\in\Sigma \\ n>0}} (\log Nv)\chi_\tau(x_v^n)\,.$$

In order to deal with L-functions of arbitrary representations we now observe that the above definitions can be extended by linearity to all virtual representations. Let

$$\tau = \sum_{\rho\in\hat{G}} n(\rho)\rho$$

be an element of the Grothendieck group of the category of representations of G; the $n(\rho)$ are integers and all but a finite number are zero. We put

$$L(\tau) = \prod_{\rho\in\hat{G}} L(\rho)^{n(\rho)}$$

and similarly

$$\frac{L'}{L}(\tau) = \sum_{\rho\in\hat{G}} n(\rho)\frac{L'}{L}(\rho)\,.$$

Let μ be a measure on the group G, which we can also consider as a measure on the space of conjugacy classes of G. For every virtual unitary representation

$$\tau = \sum_{\rho\in\hat{G}} n(\rho)\rho\,, \qquad n(\rho) = 0 \qquad \text{for almost all } \rho\,,$$

we put

$$\hat{\mu}(\tau) = \int_G \chi_\tau(g)d\mu\,,$$

where χ_τ is the trace of the representation τ. Since χ_τ is bounded, the integral converges if the total mass of $|\mu|$ is finite. The function $\tau \to \hat{\mu}(\tau)$ will be called the Fourier transform of the measure μ. In analogy with the Harmonic analysis on the group $\mathbf{R}_+^*$, it is useful to consider the integrals $\hat{\mu}(\tau)$ for τ not necessarily unitary; we then refer to $\tau \to \hat{\mu}(\tau)$ as the Fourier-Laplace transform of μ.

Definition. A not necessarily continuous function $f : G \to \mathbf{C}$ is called positive definite if for every choice of $c_1, \ldots, c_n \in \mathbf{C}$ and $g_1, \ldots, g_n \in G$ we have

$$\sum_{i,j} c_i\bar{c}_j f(g_i g_j^{-1}) \geqslant 0\,.$$

A measure μ on the group G is positive, denoted $\mu \geqslant 0$, if for every non-negative function $f : G \to \mathbf{R}_+$ we have $\int_G f(g)d\mu \geqslant 0$.

If μ is a positive measure of finite total mass, then we have for every virtual unitary representation ρ

$$\hat{\mu}(\rho\otimes\bar{\rho}) \geqslant 0 \qquad (\text{for } \mu \geqslant 0)\,.$$

In fact, since $\chi_{\rho\otimes\bar{\rho}} = |\chi_\rho|^2$ (see Part III, §1) we have

$$\hat{\mu}(\rho\otimes\bar{\rho}) = \int_G \chi_{\rho\otimes\bar{\rho}}(g)d\mu = \int_G |\chi_\rho(g)|^2\, d\mu \geqslant 0\,.$$

More generally, if $c_1, ..., c_n \in \mathbf{C}$ and $\rho_1, ..., \rho_n$ are virtual unitary representations, then we have for any positive measure μ on G with finite total mass

$$\sum_{i,j} c_c\bar{c}_j\hat{\mu}(\rho_i\otimes\bar{\rho}_j) = \int_G |\sum_{i=1}^{n} c_i\chi_{\rho_i}(g)|^2\, d\mu \geqslant 0\,.$$

For a real number $s = \sigma > 1$ and a virtual unitary representation τ, we have that the expression $\Lambda_\sigma(\tau) = -\dfrac{L'}{L}(\tau\omega_\sigma)$ is the Fourier Transform of the positive measure of finite total mass

$$\mu_\sigma = \sum_{\substack{v\in\Sigma \\ n>0}} (\log Nv)\cdot Nv^{-n\sigma}\cdot\delta[x_v^n]$$

defined on G, where $\delta[a]$ denotes the Dirac measure concentrated at a. Therefore we have, for every virtual unitary representation ρ of G and $\sigma > 1$

$$\Lambda_\sigma(\rho\otimes\bar{\rho}) = \hat{\mu}_\sigma(\rho\otimes\bar{\rho}) \geqslant 0\,.$$

Let $\tau \in \hat{G}$ and let $\nu(\tau)$ denote the order of the pole of L at $\tau\omega_1$, that is to say we write

$$L(\tau\omega_s) = \frac{\tilde{L}(\tau\omega_s)}{(s-1)^{\nu(\tau)}}\,,$$

where $\tilde{L}(\tau\omega_s)$ remains bounded and non-zero as $s \to 1$. Since

$$-\frac{L'}{L}(\tau\omega_s) = \frac{\nu(\tau)}{s-1} + f(\tau\omega_s)\,,$$

i.e. $\nu(\tau)$ is the residue of $-\dfrac{L'}{L}$ at $\tau\omega_1$, we can extend the definition of $\nu(\tau)$ by additivity to the Grothendieck group of the category of unitary representations of G. For these we have

$$\begin{aligned}\nu(\tau) &= \lim_{\sigma\to 1^+} (\sigma-1)\left(-\frac{L'}{L}(\tau\omega_\sigma) + f(\tau\omega_\sigma)\right)\\ &= \lim_{\sigma\to 1^+} (\sigma-1)\Lambda_\sigma(\tau)\,.\end{aligned}$$

Hence from the inequality $\Lambda_\sigma(\rho\otimes\bar{\rho}) \geqslant 0$ which holds true for $\sigma > 1$, we obtain, since $\sigma - 1 > 0$, that

$$\nu(\rho\otimes\bar{\rho}) \geqslant 0$$

for every virtual unitary representation ρ of G. More generally if $c_1, \ldots, c_n \in \mathbf{C}$ and $\rho_1, \ldots, \rho_n$ are virtual unitary representations, then we have

$$\sum_{i,j} c_k\bar{c}_j\bar{\nu}(\rho_i\otimes\bar{\rho}_j) \geqslant 0\,,$$

i.e. the symmetric matrix $\{\nu(\rho_i\otimes\bar{\rho}_j)\}$ is positive semi-definite.

The assumptions in the Main Theorem can now be translated into properties about the integer valued function $\nu(\tau)$. First of all the fact that $L(\tau)$ has an analytic continuation to the region $R(\tau) \geqslant 1$ and that $L(\tau)$ is holomorphic in this region except for $L(\omega_s)$ which has a simple pole at $s = 1$ implies that $\nu(\tau) \leqslant 0$ for all $\tau \neq 1$ and $\nu(1) = 1$. If $L(\tau\omega_s)$ has a zero at $s = 1$, then by conjugating the Euler product that defines $L(\tau\omega_0)$ for σ a real number, we see that $L(\bar{\tau}\omega_s)$ also has a zero at $s = 1$ of the same order as $L(\tau\omega_s)$; hence $\nu(\tau) = \nu(\bar{\tau})$. This then reduces the proof of Deligne's Theorem to the following:

MAIN LEMMA. *Let G be a locally compact group; let $\hat{G}$ be the space of irreducible unitary representations of G; consider a function*

$$\nu : \hat{G} \to \mathbf{Z}$$

that satisfies the following conditions:

a) *for the trivial representation* 1, $\nu(1) = 1$

b) $\nu(\tau) = \nu(\bar{\tau})$

c) $\nu(\tau) \leqslant 0$ *for* $\tau \neq 1$

d) $\nu(\tau\oplus\lambda) = \nu(\tau) + \nu(\lambda)$

e) $\nu(\rho\otimes\bar{\rho}) \geqslant 0$ *for every unitary representation* ρ, *i.e.* ν *is positive semi-definite.*

Then $\nu(\tau) = 0$ for all $\tau \neq 1$ except possibly for at most one τ of dimension 1 and defined by a character of order two.

§5. Reduction to the compact case: reformulation of the Main Lemma. In outline the proof of the Main Lemma is an adaptation to locally compact groups of the following argument which works for any finite group. The Plancherel theorem for a finite group G gives the decomposition of the regular representation r_G into its irreducible constituents; if χ_r is the character of r_G and χ_τ runs over the characters of the irreducible representation τ of G, then we have

$$\chi_r = \sum_{\tau \in \hat{G}} (\dim \tau)\chi_\tau .$$

Now we recall that the support of χ_r is concentrated at the identity e of G, in fact $\chi_r = |G| \delta[e]$. If we now use that $0 \leqslant \chi_r$ and evaluate the function ν which appears in the Lemma at χ_r and use the property e) we obtain

$$0 \leqslant \sum_{\tau \in \hat{G}} (\dim \tau)\nu(\tau) .$$

Properties a) and c) imply that all the terms in the above sum except $\nu(1) = 1$ are non-positive and therefore at most one other term can have $\nu(\tau) = -1$ and for this representation $\dim \tau = 1$ and $\tau = \bar{\tau}$. Hence such a τ is defined by a character of order 2. In particular, if G admits no subgroup of index two, then there is no exceptional representation.

The adaptation of the above idea consists in obtaining uniform approximations to the character of the regular representation of G by a finite linear combination with positive integer coefficients of the characters of finite dimensional irreducible unitary representations. The approximation should be fairly good so that the character of the corresponding representation is still a non-negative function. As is well known, the proper framework for the study of this type of approximation is the theory of almost periodic functions on the group G. Rather than using the full theory we shall work with an intermediary object, the Bohr Compactification G^b of G, which is a compact group. This will simplify the analysis, since on G^b we can use the full strength of the Peter-Weyl Theorem. In fact, for our purposes, even the Stone-Weierstrass approximation Theorem would suffice.

In the following we recall the basic facts about the Bohr Compactification. The reader can find an exposition of the theory in Weil [11], Chap. VII.

If $\tau : G \to GL(H_\tau)$ is an irreducible unitary linear representation, then the image of G under τ is contained in a unitary subgroup $U(H_\tau)$ of $GL(H_\tau)$; since each $U(H_\tau)$ is a compact group, their product $\prod_{\tau \in \hat{G}} U(H_\tau)$ is also a compact group. We thus obtain a map

$$\eta : G \to \prod_{\tau \in \hat{G}} U(H_\tau)$$
$$g \to (\tau(g))_{\tau \in \hat{G}} .$$

The Bohr compactification of the group G, which we denote by G^b is the closure in $\prod_{\tau \in \hat{G}} U(H_\tau)$ of the image of G under the map η. The main reason for introducing the group G^b is that it is compact and that any irreducible unitary finite dimensional representation $\tau : G \to U(H_\tau)$ factors through a finite dimensional unitary representation of G^b:

$$G \to G^b \to \prod_{\tau \in \hat{G}} U(H_\tau) \xrightarrow{pr_\tau} U(H_\tau) .$$

Now since G has a dense image in G^b, any representation of G^b is irreducible if and only if its restriction to G is irreducible. The group G^b is uniquely defined up to isomorphism by G. By projection, any unitary representation of G can be extended to G^b:

$$\begin{array}{c} \tau : G \to U(H_\tau) . \\ \downarrow \\ \mathrm{G}^b \end{array}$$

This then establishes an equivalence between the category of finite dimensional unitary representations of G^b and the category of finite dimensional unitary representations of G under which irreducible representations correspond.

More to the point at hand, which is that of obtaining good uniform approximations to the character of the regular representation of G, is the fact that the continuous functions on G^b are in one-to-one correspondence with the almost periodic functions on the locally compact group G in the sense of von Neumann.

For a locally compact abelian group G, Pontrjagin's duality theory gives very precise information about the group G^b. In fact in this case all irreducible representations of G are of dimension 1. The Pontrjagin dual of G is the group of all continuous homomorphisms

$$\hat{G} = \mathrm{Hom}_c(G, \mathbf{T}) ,$$

where $\mathbf{T} = \{z \in \mathbf{C} : |z| = 1\}$ is the circle group; furthermore $\hat{\hat{G}} = G$ and the dual of a compact group is a discrete group and vice versa. Now G^b is a compact abelian group and its character group is

$$\begin{aligned} \hat{G^b} &= \mathrm{Hom}_c(G^b, \mathbf{T}) \\ &= \mathrm{Hom}_c(G, \mathbf{T}) \\ &= \hat{G} . \end{aligned}$$

Hence G^b is the Pontrjagin dual of $\hat{G}$ viewed as a discrete group, i.e. the group of not necessarily continuous homomorphisms

$$G^b = \mathrm{Hom}_{gp}(\hat{G}, \mathbf{T}).$$

Example 1. If $G = \mathbf{R}$, then $G^b = \mathrm{Hom}_{gp}(\mathbf{R}, \mathbf{T})$, i.e. G^b is the group of all exponential functions $f(x) = e^{ixy}$. The Weierstrass Approximation Theorem describes the relation between the almost periodic functions on $\mathbf{R}$ and the continuous functions on G^b.

Example 2. If $G = \mathbf{Z}$, then $G^b = \mathrm{Hom}_{gp}(\mathbf{T}, \mathbf{T})$. The almost periodic functions on G are closely related with the trigonometric sums

$$\sum_{\lambda} c(\lambda)\chi_{\lambda}, \qquad \sum | c(\lambda) | < \infty,$$

where $\chi(n) = e^{i\lambda n}$, with real frequencies λ.

Example 3. An example relevant to the theorem at hand is $G = K \times \mathbf{R}$, the direct product of a compact group K and the group of real numbers. The Bohr compactification of G is

$$G^b = K^b \times \mathbf{R}^b.$$

In this situation the general theory shows that the class of central functions f on G with the property that if $\varepsilon > 0$, there exist a finite set of characters of unitary representations $\chi_1, ..., \chi_N$ of K and almost periodic functions $a_1, ..., a_N$ on $\mathbf{R}$ such that for all $g = (k, x)$ in G

$$| f(g) - \sum_{i=1}^{N} \chi_i(k) a_i(x) | < \varepsilon,$$

coincides with the class of central continuous functions on G^b.

Remark. After this brief interlude into the realm of almost periodic functions on the group G, the reader should keep in mind that it is quite immaterial whether we work with G or with its Bohr compactification. What is really at the heart of the argument is the family of functions F on the group G which can be uniformly approximated by finite linear combinations of the characters of irreducible unitary representations of G with complex coefficients; the structure of F can in turn be described by the Stone-Weierstrass approximation theorem.

In order to establish the Main Lemma we may then assume that G is compact. Most of Part III is devoted to the proof of the following lemma.

MAIN LEMMA (Reformulation). *Let G be a compact group; let $\hat{G}$ be the space of irreducible unitary representations of G; consider a function*

$$\nu : \hat{G} \to \mathbf{Z}$$

that satisfies the following conditions

a) *for the trivial representation* 1, $\nu(1) = 1$

b) $\nu(\tau) = \nu(\bar{\tau})$

c) $\nu(\tau) \leqslant 0$ *for* $\tau \neq 1$

d) $\nu(\tau \oplus \lambda) = \nu(\tau) + \nu(\lambda)$

e) $\nu(\rho \otimes \bar{\rho}) \geqslant 0$ *for every unitary representation* ρ, *i.e.* ν *is positive semi-definite.*

Then $\nu(\tau) = 0$ *for all* $\tau \neq 1$ *except possibly for at most one* τ_0 *of dimension* 1 *and defined by a character of order two.*

PART III: PROOF OF THE MAIN LEMMA

§1. REVIEW OF THE REPRESENTATION THEORY OF COMPACT GROUPS. We start by recalling some known facts which are standard results from the representation theory of compact groups. Some of these results are elementary, others arise in the proof or are consequences of the Peter-Weyl Theorem.

G will denote a compact topological group; G is endowed with an invariant measure $d\mu$ which we normalize so that $\int_G d\mu = 1$. An important set of functions on G is the space of square integrable functions:

$$L^2(G) = \{f : G \to \mathbf{C} \mid \int_G |f|^2 \, d\mu < \infty\}.$$

In the following we shall also consider the space of central square integrable functions on G:

$$L_c^2(G) = \{f \in L^2(G) \mid f(aga^{-1}) = f(g) \qquad \text{for all } a \in G\}.$$

Both $L^2(G)$ and $L_c^2(G)$ are Hilbert spaces with the inner product

$$(f, h) = \int_G f \cdot \bar{h} d\mu .$$

By $\hat{G}$ we denote the set of isomorphism classes of irreducible unitary representations of G. To avoid complicated notation, we shall not distinguish