# Appendix B

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# APPENDIX B

Let $K$ be a separable quadratic extension of a field $k$. We denote $x \mapsto \bar{x}$ the non trivial element $\mathrm{Gal}(K/k)$. Let $D$ be a simple algebra with dimension $n^2$ over its center $K$. We will check the criterion of the text, for the existence of an involution of the second kind on $D$, i.e. of an anti-involution * of $D$, inducing $x \mapsto \bar{x}$ on $K$. The criterion is that $N_{K/k}\,\mathrm{cl}(D) = 0$ in $\mathrm{Br}(k)$.

Let us localize, for the étale topology, over $\mathrm{Spec}(k)$. This means making large enough étale extensions of scalars $\otimes_k k'$, and keeping track of the functoriality in $k'$. The field $K$ becomes the separable quadratic extension $K' = K \otimes_k k'$ of $k'$. The algebra $D$ becomes $D' = D \otimes_k k'$, and is of the form $D' = \mathrm{End}_{K'}(V')$, for $V'$ a free module $K'$. The module $V'$ is not determined uniquely by $D'$, only up to homotheties (the corresponding projective space is uniquely determined).

For any $K$-module $M$, let $M^-$ be the module deduced from $M$ by the extension of scalars $^- : K \to K$, i.e. the module, unique up to unique isomorphism, provided with an anti-linear isomorphism $x \mapsto \bar{x} : M \xrightarrow{\sim} M^-$. Similarly for $K'$-modules. If $D' = \mathrm{End}(V')$, then $D^{-\prime} = \mathrm{End}(V'^-)$, and

$$(D \otimes_K D^-)' = \mathrm{End}(V' \otimes V'^-).$$

Let $W'$ be the fixed subspace of the anti-linear automorphism of $V' \otimes V'^-$ defined by $v \otimes \bar{w} \mapsto w \otimes \bar{v}$. It is the space of Hermitian forms on the dual of $V'$. One has $W' \otimes_{k'} K' = V' \otimes V'^-$. If $D_1 \subset D \otimes_K D^-$ is the fixed subspace of the anti-linear automorphism of $D \otimes_K D^-$ defined by $x \otimes \bar{y} \mapsto y \otimes \bar{x}$, then $D'_1$ is the $k'$-form of the $K'$-algebra $(D \otimes_K D^-)' = \mathrm{End}(V' \otimes V'^-)$ deduced from the $k'$-form $W'$ of the $K'$-module $V' \otimes V'^- : D'_1 = \mathrm{End}_{k'}(W')$.

Involutions of the second kind on $D'$ correspond one to one to non degenerate Hermitian forms on $V'$, taken up to a factor (in $k'^*$). Those, in turn, by the "dual form" construction, correspond to "non degenerate" elements of $W'$. Again, one has to take them up to a factor. The projective space $\mathbf{P}(W')$ over $k'$ is determined up to unique isomorphism by $D'$. It is hence (this is the point of localisation) defined over $k : \mathbf{P}(W') = P \otimes_k k'$, functorially in $k'$. The $k$-points of $P$ (rather, the non degenerate points) parametrize the involutions of the second kind on $D$.

The functorial isomorphism $D'_1 = \mathrm{End}_{k'}(W')$ shows that $P$ is the form of projective space (Severi-Brauer variety) attached to $D_1$. It has a rational point, and is then the ordinary projective space, if and only if $D_1$ is a matrix algebra.

This shows that $D$ has involutions of the second kind if and only if the class of $D_1$ in $\mathrm{Br}(k)$ is trivial. This class is the required norm $N_{K/k}(\mathrm{cl}(D))$. In the localization spirit, this can be deduced from the fact that the homothety by $\lambda \in K'^*$ of $V'$ induces on $W'$ the homothety by $N_{K'/k'}(\lambda) \in k'^*$.

## Appendix C

For $n \geqslant 3$, examples can be obtained as follows: take $k' = \mathbf{Q}[\zeta]$, with $\zeta = \exp(2\pi i/n)$, and $k = k' \cap \mathbf{R}$. Fix $a, b \in k^*$ and let $D$ be the $k'$-algebra generated by $X$, $Y$, subject to

$$X^n = a\,, \qquad Y^n = b$$
$$XY = \zeta YX\,.$$

It admits the anti-involution $*$, inducing complex conjugation on $k'$, defined by $\zeta^* = \zeta^{-1}$, $X^* = X$, $Y^* = Y$. The algebra $D$ is of the type we require, provided it is a division algebra. This happens already with $a, b \in \mathbf{Z}$: take for $a$ a prime congruent to 1 mod $n$, and for $b$ an integer whose residue mod $a$ has in the cyclic group of order $n$ $(\mathbf{Z}/(a))^*/(\mathbf{Z}/(a))^{*n}$ an image of exact order $n$. For instance $n = 3$, $a = 7$, $b = 2$. For $n = 2$, one proceeds similarly with $k' = \mathbf{Q}[i]$, $\zeta = -1$, $a$ congruent to 1 mod 4 and $b$ not a square mod $a$. For instance, $a = 5$, and $b = 2$. In each case, the assumption on $a$ ensures that $k'$ embed in the $a$-adic completion $\mathbf{Q}_a$ of $\mathbf{Q}$, and the fact that $D$ is a division algebra can be seen locally at $a$ : $D \otimes_{k'} \mathbf{Q}_a$ is a division algebra with center $\mathbf{Q}_a$.