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ON FREE SUBGROUPS OF SEMI-SIMPLE GROUPS

by A. BOREL

In [8], P. Deligne and D. Sullivan show that an odd-dimensional sphere $S^{2n+1} (n \geqq 1)$ admits a non-commutative free goup of isometries which acts freely. This answers a question raised in [7] (for n even, it was settled there for n odd), recalled in [11], and motivated by the fact that this property implies a strong form of the Hausdorff-Banach-Tarski paradox [6] (see §4). The present paper grew out of the attempt to extend this result to homogeneous spaces of compact semi-simple Lie groups having zero Euler characteristic. More generally we shall prove:

THEOREM A. *Let U be a non-trivial connected semi-simple compact Lie group. Then U contains a non-commutative free subgroup Γ with the following property: for any proper closed subgroup V, any element $\gamma \in \Gamma - \{1\}$, acting by left translations on U/V, has exactly $\chi(U/V)$ fixed points, where $\chi(U/V)$ is the Euler characteristic of U/V.*

In particular, Γ acts freely if $\chi(U/V) = 0$. Note that since every translation by an element of U is homotopic to the identity, the number of fixed points is the smallest possible in view of the Lefschetz fixed point theorem. The proof shows in fact that there are "many" such subgroups: given $m \in \mathbf{N}$, the set of m-tuples of elements in U which do not generate freely a free subgroup with the property mentioned in the theorem is contained in a set of Haar measure zero in U^m.

The result of [6] alluded to above also extends to actions of free groups *with commutative isotropy groups* (called "locally commutative" in [6]). This suggests looking for such actions in case $\chi(U/V) \neq 0$. We shall see indeed in §4, by a completely different argument, that they always exist (see Theorem 3). This in particular answers a question of T. J. Dekker for S^4 [6].

Let w be a reduced non-trivial word in m letters and their inverses. It defines an obvious map $f_w : U^m \to U$. The main step to prove Theorem A is to show that f_w is a dominant map. In particular Im f_w contains a non-empty open set. Furthermore, $\chi(U/V)$ can be described purely in Lie group terms, by a theorem of Hopf-Samelson [10], (recalled below). This suggests proving more general results for semi-simple algebraic groups, and deriving the above ones as special cases. We shall do so and show first

THEOREM B. *Let* G *be an algebraic connected semi-simple group. Let* $m \in \mathbf{N}$, *and* $f_w : G^m \to G$ *be the map associated to a non-trivial element* w *in the free group on* m *letters. Then* $\operatorname{Im} f_w$ *is not contained in any proper subvariety* ($m \geqslant 2$).

The proof is by induction on dim G and uses a variant of the key idea of [8]. In this statement, we have implicitly identified an algebraic group H with the group $H(\Omega)$ of its points in a "universal field" Ω, i.e., an algebraically closed extension of infinite transcendence degree of a prime field. Assume now that G is defined over a field K of infinite transcendence degree. It follows from [17], and was well-known over **R** or **C**, that $G(K)$ contains many non-commutative free subgroups, in fact that m "sufficiently general" elements are free generators of a subgroup ($m \in \mathbf{N}$). Theorem B implies a sharpening of that assertion, namely the existence of non-commutative free subgroups in $G(K)$ all of whose elements, except the identity, are outside a given proper subvariety (or even outside a countable union of proper subvarieties defined over a common field of finite transcendence degree over the prime field, see Theorem 2 for the precise statement). As an application, we deduce

THEOREM C. *Let* K *be a field of infinite transcendence degree over its prime field and assume* G *to be defined over* K. *Then there exists* $g = (g_i) \in G(K)^m$ *whose components* g_i *freely generate a subgroup* Γ *of* $G(K)$ *such that every* $\gamma \in \Gamma - \{1\}$ *is regular and generates a Zariski-dense subgroup of the unique maximal torus* T_γ *containing it.*

In fact, there are many such g's. In some sense, a "generic" $g \in G(K)^m$ always gives rise to such a subgroup. If $K = \mathbf{R}, \mathbf{C}$ or is a p-adic field, the set of such g's is dense in the ordinary topology.

Given a closed subgroup H of G, set $\chi(G, H) = 0$ if H does not contain any maximal torus of G. If it does, and T is one, then set $\chi(G, H) = [N_G(T) : N_H(T)]$. Then Theorem C implies the:

COROLLARY. *Every* $\gamma \in \Gamma - \{1\}$, *acting by left translations on* $G(K)/H(K)$, *has at most* $\chi(G, H)$ *fixed points.*

In particular, Γ acts freely if H does not contain any maximal torus of G. Assume now that it contains one, say T_0, which we may assume to be defined over K. Assume further that all maximal K-tori of H are conjugate under $H(K)$ and set

$$\chi(G(K), H(K)) = [N_{G(K)}(T_0) : N_{H(K)}(T_0))] \,.$$

Then we shall see that γ acts freely on $G(K)/H(K)$ if T_γ is not conjugate to T_0 under $G(K)$ and has $\chi(G(K), H(K))$ fixed points otherwise.

The condition on H and the second alternative hold either if K is algebraically closed or if $K = \mathbf{R}$ and $G(K)$ is compact. In that last case, $\chi(G(K), H(K)) = \chi(G(K)/H(K))$ by [10], and Theorem A follows.

I wish to thank D. Sullivan for having sent me a preprint of [8], which was the starting point of the present paper, and D. Kazdhan and G. Prasad for having pointed out two errors in a previous proof of Theorem B for $\mathbf{SL}_n$.

Notation and conventions. In the sequel, G is a connected semi-simple algebraic group over some groundfield, and p the characteristic of the groundfield. For unexplained notation and notions on linear algebraic groups, we refer to [1]. In particular, in such a group, the word "torus" is meant as in [1], i.e., refers to a connected linear algebraic group which is isomorphic to a product of $\mathbf{GL}_1$'s. In a compact group however it means a topological torus (product of circle groups).

If H is a group, and A, B are subsets of H, then

$$^{B}A = \{bab^{-1} \mid a \in A, b \in B\},\ N_H(A) = \{h \in H \mid hAh^{-1} = A\}\,,$$
$$\mathrm{Tr}_H(A, B) = \{h \in \mathrm{H} \mid h.A.h^{-1} = B\}\,.$$

If Γ acts on a space X, the *isotropy group* of Γ at x is

$$\Gamma_x = \{\gamma \in \Gamma \mid \gamma \cdot x = x\}\,.$$

We recall that a morphism $f : X \to Y$ of irreducible algebraic varieties is *dominant* if its image is not contained in any proper algebraic subvariety. If so, then Im f contains a Zariski-dense open subset of Y [1: AG 10.2]. If the groundfield has characteristic zero, then, since f is separable, the differential of f has maximal rank on some non-empty Zariski open subset of X [1: AG, 17.3].

§1. Proof of Theorem B

Let m be an integer $\geqq 2$. Let $w = w(X_1, ..., X_m)$ be a non-trivial element in the free group $F(X_1, ..., X_m)$ on m letters X_i, i.e., a non-trivial reduced word in the X_i's, with non-zero integral exponents [3: I.81, Prop. 7]. Then given a group H, the word w defines a map $f_w : H^m \to H$ by the rule

$$(1) \qquad f_w(\{h_1, ..., h_m\}) = w(h_1, ..., h_m)\,, \qquad (h_i \in H\,;\ 1 \leqq i \leqq m)\,.$$

If H is an algebraic group, then f_w is a morphism of algebraic varieties which is defined over any field of definition for H. In the case where $H = G$ we want to prove

THEOREM 1. *The map* $f_w : G^m \to G$ *is dominant.*

This is a geometric statement. To prove it, we shall identify G with $G(\Omega)$, where Ω is some universal field. We have then to prove that $f_w(G(\Omega)^m)$ is Zariski-dense in $G(\Omega)$.

The Zariski closure Z of $\mathrm{Im} f_w$ is irreducible (since G^m is) and is invariant under conjugation, since $\mathrm{Im} f_w$ is obviously so. Since the semi-simple elements of G are Zariski-dense, and all conjugate to elements in some fixed maximal torus T, it suffices to show that $Z \supset T$.

a) We first consider the case where $G = \mathbf{SL}_n (n \geqq 2)$. Let us prove that $G(\Omega)$ contains a Zariski-dense subgroup H, no element of which, except for the identity, has an eigenvalue equal to one. This statement and its proof were directly suggested by [8].

One can find an infinite field L of the same characteristic as Ω over which there exists a central division algebra D of degree n^2. We may for example take for L a local field (see e.g. XIII, §3, Remarque p. 202 in [14]). We may assume $L \subset \Omega$. Let $\mathscr{D}^1$ be the algebraic group over L whose points in a commutative L-algebra M are the elements of reduced norm one in $D \otimes_L M$. Then $\mathscr{D}^1$ is an anisotropic L-form of $\mathbf{SL}_n$. Of course, D splits over Ω and the isomorphism $D \otimes_L \Omega = \mathbf{M}_n(\Omega)$ yields an isomorphism of $\mathscr{D}^1(\Omega)$ onto $G(\Omega)$. We let H be the image of $D^1 = \mathscr{D}^1(L)$ under such an isomorphism. The group H is Zariski-dense since L is infinite. The fact that any $h \in H - \{1\}$ has no eigenvalue equal to one is then proved as in [8]: the element $h - 1$ is a non-zero element of D, hence is invertible, hence has no eigenvalue zero and therefore h has no eigenvalue one. This proves our assertion. Let p_0 be the characteristic exponent of Ω ($p_0 = 1$ if char $\Omega = 0$ and $p_0 =$ char Ω otherwise). If $p_0 = 1$, then H consists of semi-simple elements; if not, then $h^q (q = p_0^{n-1})$ is semi-simple for any $h \in G$. Let $f_w^q : G^m \to G$ be defined by $f_w^q(g) = f_w(g)^q$. Then $f_w^q(H)$ consists of semi-simple elements. Let Z_q be the Zariski closure of $\mathrm{Im} f_w^q$. Since $x \mapsto x^q$ is dominant, we have shown:

(*) *Let* V *be the set of semi-simple elements in* $G(\Omega)$ *which have no eigenvalue equal to one. Then* $\{1\} \cup (V \cap \mathrm{Im} f_w^q)$ *is Zariski-dense in* Z_q.

We now prove the theorem for $\mathbf{SL}_n (n \geqq 2)$ by induction on n. It suffices to show that f_w^q is dominant and, for this, that $Z_q \supset T$. Let $n = 2$. The group $\mathbf{SL}_2$ has dimension three and the conjugacy classes of non-central elements have dimension two. If $Z_q \neq G$, then $\dim Z_q \leqq 2$ and Z_q is contained in the union of the set U of unipotent elements of G and of finitely many conjugacy classes of semi-simple elements $\neq 1$. Those are closed, disjoint from U. Since Z_q is irreducible and contains 1, it should then be contained in U. On the other hand,

$Z_q \neq \{1\}$ since G contains non-commutative free subgroups, as follows from [17] (see also Remark 1 below). We then get

$$\{1\} \underset{\neq}{\subset} Z_q \subset U\,,$$

but this contradicts (*), whence the Theorem for $\mathbf{SL}_2$.

Assume now $n > 2$ and our assertion proved up to $n - 1$. This implies in particular that Z_q contains all subgroups of G isomorphic to $\mathbf{SL}_{n-1}$, hence that $Z_q \cap T$ contains the subtori of T of codimension one consisting of the elements of T which have at least one eigenvalue equal to one. Call Y their union. Assume that $Z_q \cap T \neq T$. Then we may write $Z_q \cap T = Y \cup Y'$, where Y' is a proper algebraic subset of T not containing any irreducible component of Y. Let Q be the Zariski-closure of the set ${}^G Y'$ of conjugates of elements of Y'. We claim that $Y \not\subset Q$. In fact, the subsets Y and Y' are stable under the Weyl group $W = N(T)/T$ (which may be identified with the group of permutations of the basic vectors of Ω^n). Let $J \subset \Omega[T/W]$ be the ideal of Y'. The algebra $\Omega[T/W]$ is isomorphic, under the restriction mapping, to the algebra S of regular class functions on G [16]. Let J' be the ideal of S corresponding to J under this isomorphism and R the variety of zeroes of J'. We have then $Q \subset R$, but $Y \not\subset R$, whence $Y \not\subset Q$.

The difference $Y' - (Y \cap Y')$ contains a conjugate of every semi-simple element of Z_q not having any eigenvalue equal to one. Therefore (*) implies that $Z_q = \{1\} \cup Q$. But this contradicts the fact that $Y \not\subset Q$. Therefore $T \subset Z_q$ and the theorem is proved for $\mathbf{SL}_n$.

b) In the general case we use induction on dim G. If $\mu : G' \to G$ is an isogeny, then the theorem for G' implies it for G, hence we may assume G to be simply connected. It is then a direct product of almost simple groups, whence also a reduction to the case where G is almost simple. By a), it suffices to consider the case where G is not isomorphic to $\mathbf{SL}_n$ for any n. But then it contains a proper connected semi-simple subgroup H of maximal rank (see lemma below). By induction Z contains a maximal torus of H, hence one of G, and therefore T.

We have just used the following lemma:

LEMMA 1. *Assume G to be almost simple, and not isogeneous to $\mathbf{SL}_n$ for any n. Then G contains a proper connected semi-simple subgroup of maximal rank.*

For convenience, we may assume G to be isomorphic to its adjoint group. Let $\Phi = \Phi(G, T)$ be the root system of G with respect to T and $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ a basis of Φ. Since G is adjoint, Δ is also a basis of the group $X^*(T)$ of rational characters of T. Let d be the dominant root and write

$$d = \sum_{i=1}^{i=l} d_i \alpha_i .$$

The d_i's are strictly positive integers. By assumption, Φ is not of type $\mathbf{A}_m$ for any m. Therefore, by the classification of root systems, one of the d_i's is prime (see e.g. [4]). Say $d_1 = q$, with q prime. Let Ψ be the set of elements in Φ which, when expressed as linear combination of simple roots, have either 0 or $\pm q$ as coefficient of α_1. This is a closed set of roots. In fact, it is a root system with basis $\alpha_2, \ldots, \alpha_l$ and $-d$ [2]. We claim that there exists a closed connected subgroup H of G containing T with root system Ψ.

Let first $q \neq$ char. K. Then there is an element $t \in T$, $t \neq 1$, such that

$$d(t) = \alpha_i(t) = 1 , \qquad (i = 2, \ldots, l) .$$

It has order q, and Ψ is the set of roots which are equal to one on t. Then the identity component of the centralizer of t satisfies our condition.

Let now $q =$ char. Ω. Let $\mathfrak{t}$ be the Lie algebra of T and $\mathfrak{u}$ be the subspace of $\mathfrak{t}$ which annihilates the differentials $d\alpha_i$ of the roots α_i $(i = 2, \ldots, l)$. It is one dimensional and does not annihilate $d\alpha_1$ (since, as recalled above, Δ is a basis of $X^*(T)$, hence the $d\alpha_i (1 \leqq i \leqq l)$ form a basis of the dual space to $\mathfrak{t}$). Of course, the differential of any $\lambda \in X^*(T)$ which is divisible by q in $X^*(T)$ is identically zero on $\mathfrak{t}$. It follows then that

$$\Psi = \{\alpha \in \Phi \mid d\alpha(\mathfrak{u}) = 0\} .$$

Let $\mathfrak{g}$ be the Lie algebra of G and

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid \mathrm{Ad}\, t(x) = \alpha(t) \cdot x (t \in T)\} , \qquad (\alpha \in \Phi) ,$$

be the (1-dimensional) eigenspace of T corresponding to α[1, §14]. The previous relation implies that

$$\mathfrak{z}_\mathfrak{g}(\mathfrak{u}) = \mathfrak{t} \oplus \bigoplus_{\alpha \in \psi} \mathfrak{g}_\alpha .$$

By [1: §14] the Lie algebra of the centralizer

$$Z_G(\mathfrak{u}) = \{g \in G \mid \mathrm{Ad}\, g(x) = x, (x \in \mathfrak{u})\} ,$$

of $\mathfrak{u}$ in G is equal to $\mathfrak{z}_\mathfrak{g}(\mathfrak{u})$; therefore $Z_G(\mathfrak{u})$ is a semi-simple subgroup satisfying our conditions.

Remarks. 1) We have used [17] only for $\mathbf{SL}_2(\Omega)$, but it is possible to bypass [17] in this case and make our proof, and the whole paper, independent of [17].

We need only to prove that $\mathbf{SL}_2(\Omega)$ contains a non-commutative free subgroup F. If Ω has characteristic zero, we may take any torsion-free subgroup of $\mathbf{SL}_2(\mathbf{Z})$. Let now $p = \operatorname{char} \Omega$ be >0. Then, by the arithmetic method, using division quaternion algebras over global fields, we can construct a discrete cocompact subgroup of $\mathbf{SL}_2(L)$, where L is a local field of characteristic p (cf. A. Borel-G. Harder, *Crelle J. 298* (1978), 53-74). The latter has a torsion-free subgroup F of finite index (H. Garland, *Annals of Math.* 97 (1973), 375-423) which is then free, since it acts freely on a tree, namely the Bruhat-Tits building of $\mathbf{SL}_2(L)$.

2) For any non-zero $n \in \mathbf{Z}$, the power map $g \mapsto g^n$ is dominant (because it is surjective on any maximal torus [1 : 8.9]), hence Theorem 1 is obvious if the sum of the exponents of one letter in the word w is not zero. (See [11] for a similar remark in the context of compact groups.)

3) If U and V are non-empty open subsets in a connected algebraic group H, then $H = U \cdot V$ [1: 1.3]. It follows then from Theorem 1 that if w, w' are two words in two letters, say, then the map $G^4 \to G$ defined by

$$f(g_1, g_2, g_3, g_4) = w(g_1, g_2) \cdot w'(g_3, g_4),$$

is surjective. For instance, every element of $G(\Omega)$ is the product of two commutators. However, the map f_w itself is not always surjective; for instance $x \mapsto x^2$ is not surjective in $\mathbf{SL}_2(\mathbf{C})$, as pointed out in [11].

4) If $K = \mathbf{C}$, then Theorem 1 implies that Im f_w contains a dense open set in the ordinary topology. If G is defined over $\mathbf{R}$, then Theorem 1 also shows that $f_w(G(\mathbf{R}))$ contains a non-empty subset of $G(\mathbf{R})$ which is open in the ordinary topology. However it may not be dense. For instance, it is pointed out in [11] that for $\mathbf{SU}_2$, the image of the map defined by $[x^2, yxy^{-1}]$ omits a neighborhood of -1; however this map is surjective in $\mathbf{SO}_3$.

It seems that little is known about the image of f_w, even over $\mathbf{R}$ or $\mathbf{C}$. A general fact however is that the commutator map is surjective in any compact connected semi-simple Lie group [9].

§2. Free subgroups with strongly regular elements

1. In the sequel, K is a field of infinite transcendence degree over its prime field. We shall need the following lemma:

LEMMA 2. *Let X be an irreducible unirational K-variety. Let L be a finitely generated subfield of K containing a field of definition of X, and $V_i (i \in \mathbf{N})$ a sequence of proper irreducible algebraic subsets of X defined over an* *algebraic closure* $\bar{L}$ *of* L. *Then* $X(K)$ *is not contained in the union of the* $V_i \cap X(K)$, $(i \in \mathbf{N})$.

By definition of unirationality, there exists for some $n \in \mathbf{N}$ a dominant K-morphism $\varphi : \mathbf{A}^n \to X$, where $\mathbf{A}^n$ denotes the affine n-dimensional space.

This map is already defined over some finitely generated extension of L. Replacing L by the former, we may assume φ to be defined over L, hence $\varphi^{-1}(V_i)$ to be defined over $\bar{L}$. It is a proper algebraic subset since φ is dominant. This reduces us to the case where $X = \mathbf{A}^n$. But then any point whose coordinates generate over $\bar{L}$ a field of transcendence degree n will do.

THEOREM 2. *Assume* G *to be defined over* K. *Let* $\mathscr{V} = \{V_i\}$ $(i \in \mathbf{N})$ *be a family of proper subvarieties of* G, *all defined over an algebraic closure* $\bar{L}$ *of a finitely generated subfield* L *of* K *over which* G *is also defined. Then* $G(K)$ *contains a non-commutative free subgroup* Γ *such that no element of* $\Gamma - \{1\}$ *is contained in any of the* V_i*'s. Given* $m \geqq 2$, *the set of* m*-tuples which freely generate a subgroup having this property is Zariski dense in* G^m.

We may (and do) assume that the identity element is contained in one of the V_i's.

Let w and f_w be as in §1. Then f_w is defined over L hence $f_w^{-1}(Z)$ is defined over $\bar{L}$ for every $Z \in \mathscr{V}$ and is a *proper* algebraic subset by Theorem 1. The sets $f_w^{-1}(Z)$, as w runs through all the non-trivial reduced words (in m letters and their inverses) and Z through $\mathscr{V}$, form then a countable collection of proper algebraic subsets, all defined over $\bar{L}$. But G, hence G^m, is a unirational variety over any field of definition of G [1: 18.2]. Lemma 2 implies therefore the existence of $g = (g_i) \in G(K)^m$ not belonging to any of these subsets. Then the g_i's are free generators of a subgroup which satisfies our conditions. In fact, we see that we can take for g any point of $G(K)^m$ which is generic over $\bar{L}$ and, since $\bar{L}$ has finite transcendence degree over the prime field, such points are Zariski-dense. This establishes the second assertion.

Remark. If $G = \mathbf{SO}_{2n}$ (resp. $\mathbf{SO}_{2n+1}$), this shows for instance the existence of a free subgroup Γ, no element of which except 1 has the eigenvalue 1 (resp. the eigenvalue 1 with multiplicity >1).

2. Any semi-simple element x of G is contained in a maximal torus [1: 11.10]; x is called regular if it is contained in exactly one maximal torus. We shall say that x is *strongly regular* if it is not contained in any non-maximal torus, i.e., if the cyclic group generated by x is Zariski-dense in a maximal torus.

The following result contains Theorem C of the introduction.

COROLLARY 1. *Assume G to be defined over K. Then $G(K)$ contains a non-commutative free subgroup Γ all of whose elements $\neq 1$ are strongly regular. Given $m \geqq 2$, the set of m-tuples $(g_i) \in G(K)^m$ which generate freely a subgroup with that property is Zariski dense in G^m.*

The field K contains a field of definition L of G which is finitely generated over its prime field. Let $\bar{L}$ be an algebraic closure of L in our universal field Ω. Then the subfield generated by $\bar{L}$ and K has infinite transcendence degree over $\bar{L}$. Let S be the set of singular elements of G (i.e., of elements $g \in G$ such that Ad g has the eigenvalue one with multiplicity $>$ rk G). It is algebraic, defined over $\bar{L}$. Fix a maximal L-torus T of G [1: 18.2]. Every proper closed subgroup of T is contained in the kernel of a rational character [1: 8.2]. The characters are all defined over a finite separable extension L' of L [1: 8.11] and form a countable set. For $\lambda \in X^*(T)$, $\lambda \neq 1$, let $T_\lambda = \ker \lambda$, and V_λ the Zariski-closure of ${}^G T_\lambda$. The V_λ and S form a countable set $\mathscr{V}$ of proper algebraic subsets of G which are all defined over $\bar{L}$.

Our assertion is now a special case of the Theorem.

3. We can now prove the Corollary in the introduction. Let Ω be an algebraically closed extension of K. Since $G(K)/H(K)$ may be identified to an orbit of $G(K)$ in $G(\Omega)/H(\Omega)$ it suffices to show:

COROLLARY 2. *Assume K to be algebraically closed. Then every $\gamma \in \Gamma - \{1\}$, operating by left translations on $G(K)/H(K)$, has exactly $\chi(G, H)$ fixed points.*

For $\gamma \in \Gamma - \{1\}$, let F_γ be the fixed point set of γ in $G(K)/H(K)$, and let T_γ be the maximal torus in which the cyclic group generated by γ is dense. Clearly, F_γ is also the set of fixed points of $T_\gamma(K)$. Thus, if F_γ is non-empty, then T_γ is conjugate to a subgroup of H, and H has maximal rank. Assume this is the case and let T_0 be a maximal K-torus of H. Since the maximal tori of H (or G) are conjugate, it is elementary that F_γ may be identified with $\mathrm{Tr}(T_0, T_\gamma)/N_H(T_0)$. But, if $x \in \mathrm{Tr}(T_0, T_\gamma)$, then $\mathrm{Tr}(T_0, T_\gamma) = x \cdot N_G(T_0)$, whence the Corollary.

4. We now generalize slightly the Corollary in case H contains a maximal torus of G, dropping again the assumption that K is algebraically closed. Assume instead

(*) *The maximal K-tori of H are conjugate under $H(K)$.*
If T_0 is a maximal K-torus of H, we then set

$$\chi(G(K), H(K)) = [N_{G(K)}(T_0) : N_{H(K)}(T_0)] .$$

If K is algebraically closed, then (*) is fulfilled and $\chi(G(K), H(K))$ is our previous $\chi(G, H)$. We again set $\chi(G(K), H(K)) = 0$ if H does not contain any maximal torus of G.

COROLLARY 3. *Let* Γ *be as in Theorem 2. Let* H *be a closed* K-*subgroup of maximal rank and assume* (*) *to be satisfied. Then* $\gamma \in \Gamma - \{1\}$ *acts freely if* T_γ *is not conjugate under* $G(K)$ *to* T_0 *and has* $\chi(G(K), H(K))$ *fixed points otherwise.*

The argument is the same as before: F_γ is also the set of fixed points of T_γ. The latter is defined over K. If $F_\gamma \neq \emptyset$, then there exists $x \in G(K)$ such that ${}^xT_\gamma \in H$, hence by (*),

$$\mathrm{Tr}_{G(K)}(T_0, T_\gamma) \neq \emptyset ,$$

and we have, as above, bijections

$$F_\gamma = \mathrm{Tr}_{G(K)}(T_0, T_\gamma)/N_{H(K)}(T_0) = N_{G(K)}(T_0)/N_{H(K)}(T_0) .$$

5. (i) If $K = \mathbf{R}, \mathbf{C}$ or also is a non-archimedean local field with finite residue field, then $G(K)$, endowed with the topology stemming from K, is a Lie group over K, and in particular is a locally compact topological group. In that case, we can use in Theorem 2 a category argument instead of Lemma 2: the $f_w^{-1}(Z)$, being proper algebraic subsets, have no interior point, the intersection of their complement is then dense by Baire's theorem, whence the last assertion of Theorem 2 with "Zariski-dense" replaced by "dense in the K-topology".

(ii) In [4] it is asked whether the hyperbolic n-space admits a non-commutative free group of isometries which acts freely. More generally, one has the

PROPOSITION. *Let* S *be a connected semi-simple non-compact Lie group with finite center,* U *a maximal compact subgroup of* L *and* $X = L/U$ *the symmetric space of non-compact type of* S. *Then* S *contains a non-commutative free subgroup which acts freely on* X.

If $\mathrm{rk}\, S \neq \mathrm{rk}\, U$, this could be deduced from Corollary 2. However, the existence of one such subgroup can be proved much more directly in all cases: if $S = \mathbf{SL}_2(\mathbf{R})$ or $\mathbf{PSL}_2(\mathbf{R})$, then we may take for Γ a free subgroup of finite index in $\mathbf{SL}_2(\mathbf{Z})$ or $\mathbf{SL}_2(\mathbf{Z})/\{\pm 1\}$. If S is of dimension >3, then it contains a copy of $\mathbf{SL}_2(\mathbf{R})$ or of $\mathbf{PSL}_2(\mathbf{R})$, and therefore a *discrete* non-commutative free subgroup Γ. No element $\gamma \in \Gamma - \{1\}$ is contained in a compact subgroup of S, hence Γ acts freely on X.

A similar argument would be valid over a non-archimedean local field K for the Bruhat-Tits buildings attached to semi-simple K-groups.

§3. Compact groups. Proof of Theorem A.

1. Let U be a compact Lie group. Then we may view U as the group $G(\mathbf{R})$ of real points of an algebraic group G defined over $\mathbf{R}$ [5]. Furthermore, the maximal (topological) tori of U are the groups $T(\mathbf{R})$, where T runs through the maximal $\mathbf{R}$-tori of G. They are conjugate under inner automorphisms of U. Corollary 1 to Theorem 2 insures the existence of a non-commutative free subgroup Γ of U such that every $\gamma \in \Gamma - \{1\}$ is strongly regular, i.e., generates a dense subgroup of a maximal torus of U. If now V is a closed subgroup of U, then, by [10], $\chi(U/V) = 0$ if V does not contain a maximal torus of U, and is equal to $[N_U(T) : N_V(T)]$ if V contains a maximal torus T of U. By the results just recalled, we may write $V = H(\mathbf{R})$, where H is an algebraic $\mathbf{R}$-subgroup of G, the condition (*) of §2 is satisfied, and any maximal torus of U is conjugate to T. Theorem A now follows from Corollaries 1 and 3 to Theorem 2.

2. The results of this paper, specialized to compact Lie groups, can of course be proved more directly, in the framework of the theory of compact Lie groups, without recourse to the theory of algebraic groups. For the benefit of the reader mainly interested in that case, we sketch how to modify the above arguments.

The main point is again to prove Theorem 1, where now G stands for a non-trivial compact connected semi-simple Lie group. In part a) of the proof, the role of $\mathbf{SL}_n$ is taken by $\mathbf{SU}_n$. If $n = 2$, G contains non-commutative free subgroups. If $n > 2$, the argument is the same except that now we take for D, exactly as in [8], a division algebra with an involution of the second kind and identify $\mathbf{SU}_n$ to $(D \otimes_L \mathbf{R})^1$, where L is the fixed field, in the center of D, of the given involution of D. In part b), we use the fact that if G is simple, not locally isomorphic to $\mathbf{SU}_n$, then it contains a proper closed connected semi-simple subgroup of maximal rank, for which we can refer directly to [2] (the proof of Lemma 1 was in fact just an adaptation to algebraic groups of the one in [2]).

Then, as pointed out in section 5 of §2, a simple category argument yields Theorem 2, whence also Corollary 1 to Theorem 2 and Theorem A.

§4. Free group actions with commutative isotropy groups

1. Let Γ be a non-commutative free group acting on a set X. Assume that Γ acts freely, or more generally, that the isotropy groups $\Gamma_x (x \in X)$ are commutative (hence cyclic), and that at least one is reduced to $\{1\}$. Then the decomposition theorem 2.2.1, 2.2.2 of [6] implies in particular the following: given $n \geqq 2$, there exists a partition of X into $2n$ subsets X_i and elements $\gamma_i \in \Gamma (1 \leqq i \leqq 2n)$ such that X is the disjoint union of $\gamma_i X_i$ and $\gamma_{n+i} X_{n+i} (i \leqq i \leqq n)$. If we view the operations of

Γ as congruences, this shows that X is equivalent to the union of n copies of itself via finite congruences. The existence of such partitions of $\mathbf{S}^2$ was proved first by R. M. Robinson [13].

This then leads to the problem of finding actions of free groups with commutative isotropy groups in cases where free actions are ruled out. We now prove some results pertaining to that question.

2. Consider first the case of $\mathbf{S}^n = \mathbf{SO}_{n+1}/\mathbf{SO}_n$. The problem is then to find a free non-commutative subgroup Γ of $\mathbf{SO}_{n+1}$ such that no two non-commutative elements of Γ are contained in a conjugate of $\mathbf{SO}_n$, i.e., have a common non-zero fixed vector. In [6], this is shown for $n \geqq 2$, but $n \neq 4$. We want to give an alternate proof which also covers that last case. For n odd, there is even a Γ such that no element $\neq 1$ has an eigenvector, as follows from the remark to Theorem 2. So assume n even. If $n = 2$, then the isotropy groups of $\mathbf{SO}_3$ itself on $\mathbf{S}^2$ are commutative, hence any non-commutative free subgroup will do. Assume $n > 2$. The group $\mathbf{SO}_3$ has an (absolutely) irreducible real representation of degree $n + 1$; it can e.g. be realized in the space of spherical harmonics in $\mathbf{R}^3$ of degree $n/2$. Let H be the image of $\mathbf{SO}_3$ in $\mathbf{SO}_{n+1}$ under such a representation and let Γ be a free non-commutative subgroup of H. Then any two non-commuting elements of Γ generate a dense subgroup of H, hence do not have a common non-zero proper invariant subspace of $\mathbf{R}^{n+1}$; in particular they have no common fixed vector, whence our assertion.

Example. For the sake of definiteness, we indicate one explicit example in the case $n = 4$.

Let $\alpha, \beta \in (0, 2\pi)$ be two angles such that the rotations of angle α and β of $\mathbf{R}^3$ around two perpendicular axes freely generate a free subgroup $F_{\alpha,\beta}$ of $\mathbf{SO}_3$. We may take e.g. $\alpha = \beta$, where α is such that $\cos \alpha$ is transcendental [7]. Let $\{e_1, ..., e_5\}$ be the canonical basis of $\mathbf{R}^5$. Let $A_\alpha \in \mathbf{SO}_5$ be the transformation which is a rotation of angle 2α in the plane $[e_4, e_5]$ spanned by e_4 and e_5 and which is the rotation of angle 4α around the axis spanned by $(3^{1/2}, 0, 1)$ in $[e_1, e_2, e_3]$. Let B_β the element of $\mathbf{SO}_5$ which fixes e_3 and is a rotation of angle 2β (resp. 4β) in the plane $[e_2, e_4]$ (resp. $[e_1, e_5]$). Then A_α and B_β freely generate an irreducible subgroup of $\mathbf{SO}_5$, whose closure is isomorphic to $\mathbf{SO}_3$ and which is therefore locally commutative on $\mathbf{S}^4$.

In fact, in suitable coordinates, this group is just the image of the group $F_{\alpha,\beta}$ under the five-dimensional irreducible representation of $\mathbf{SO}_3$. The easy computations showing this are left to the reader.

3. The above argument extends in the general case to the following sharpening of Theorem A in the case of non-zero Euler characteristic.

THEOREM 3. *Let U be a compact connected non-trivial semi-simple Lie group. Then U contains a non-commutative free subgroup Γ whose elements $\gamma \neq 1$ are regular and such that, for any proper closed subgroup V of maximal rank of U, the isotropy groups $\Gamma_x (x \in U/V)$ of Γ on U/V are commutative and any $\gamma \in \Gamma - \{1\}$ has exactly $\chi(U/V)$ fixed points.*

Proof: First we carry an easy reduction to the case where U is simple and V connected. Let U' be the quotient of U by its center, $\pi : U \to U'$ the natural projection and $V' = \pi(V)$. The isotropy groups of U on U'/V' contain the isotropy groups on U/V, hence we may assume that U has center reduced to the identity. Let V^0 be the identity component of V. Any isotropy group of Γ on U/V contains an isotropy group on U/V^0 as a subgroup of finite index. Both are therefore simultaneously commutative or not commutative. So we may assume V to be connected. Now U is a direct product of simple groups and V, being of maximal rank, is the direct product of its intersections with the simple factors of U [2], whence our reduction.

We now prove the theorem in this case except for the last assertion on the number of fixed points.

If $U = \mathbf{SO}_3$, then any proper closed subgroup has a commutative subgroup of finite index, and any element $\neq 1$ is regular. Therefore we may take for Γ any non-commutative free subgroup. Assume now that $U \neq \mathbf{SO}_3$, hence dim $U > 3$. Then U has a closed subgroup H, isomorphic to $\mathbf{SO}_3$, which contains regular elements of U and is not contained in any proper subgroup of maximal rank [15 : §12]. (This subgroup is called "principal" in [15].) Then any element of infinite order in H is regular in U. In particular any element $\gamma \neq 1$ in a free non-commutative subgroup Γ of H is regular. Moreover any two non-commuting elements of Γ generate a dense subgroup of H. If they were contained in a conjugate of V, then so would H, whence a contradiction.

There remains to see that every $\gamma \in \Gamma - \{1\}$ has exactly $\chi(U/V)$ fixed points on U/V. Let S_γ be the closure of the subgroup of H generated by γ. It is a one-dimensional torus, almost all of whose elements are regular in U. Fix a maximal torus T_0 of V, hence of U. If $x, y \in U$ are such that ${}^xS, {}^yS \subset T_0$, then the inner automorphism by $x \cdot y^{-1}$, which brings ${}^yS_\gamma$ onto ${}^xS_\gamma$, must leave T_0 stable since ${}^xS_\gamma$ contains regular elements, i.e., $x \cdot y^{-1} \in N_U(T_0)$. From this we see again that there is a natural bijection between the fixed point set of γ and $N_U(T_0)/N_V(T_0)$, and our assertion follows as in section 4 of §2.

4. The same argument is valid for a complex semi-simple Lie group, using a principal three-dimensional subgroup, or also over any algebraically closed groundfield. Over a field K of infinite transcendence degree over its prime field,

one would have to assume the existence of a principal three-dimensional subgroup which is defined over K.

5. We note finally that if $\Gamma \subset G(K)$ satisfies the conditions of Corollary 1 to Theorem 2 and if H is a subgroup of maximal rank of G whose identity component is solvable, then for any $x \in G(K)/H(K)$, the isotropy group Γ_x is commutative, since its intersection with the isotropy group of x in $G(K)$ is on one hand free, as a subgroup of Γ, and on the other hand contains a solvable normal subgroup of finite index, since $H(K)$ does.

REFERENCES

[1] BOREL, A. *Linear Algebraic Groups.* (Notes by H. Bass), Benjamin, New York 1969.
[2] BOREL, A. et J. DE SIEBENTHAL. Les sous-groupes fermés de rang maximum des groupes de Lie clos. *Comm. Math. Helv. 23* (1949), 200-220.
[3] BOURBAKI, N. *Algèbre 1, 2, 3.* Hermann, Paris 1970.
[4] —— *Groupes et algèbres de Lie, Chap. 4, 5, 6.* Masson, Paris 1981.
[5] CHEVALLEY, C. *Théorie des Groupes de Lie III : Groupes algébriques.* Hermann, Paris 1955.
[6] DEKKER, T. J. Decompositions of sets and spaces I, II, III. *Indag. Math. 18* (1956), 581-595, *19* (1957), 104-107.
[7] —— On free groups of motions without fixed points. *ibid. 20* (1958), 348-353.
[8] DELIGNE, P. and D. SULLIVAN. Division algebras and the Hausdorff-Banach-Tarski paradox. *L'Enseignement Mathématique 29* (1983), 145-150.
[9] GOTO, M. A theorem on compact semi-simple groups. *Jour. Math. Soc. Japan 1* (1949), 270-272.
[10] HOPF, H. und H. SAMELSON. Ein Satz über die Wirkungsräume geschlossener Liescher Gruppen. *Comm. Math. Helv. 13* (1940), 240-251.
[11] MYCIELSKI, J. Can one solve equations in groups? *Amer. Math. Monthly 84* (1977), 723-726.
[12] —— Equations unsolvable in $GL_2(C)$ and related problems. *ibid. 85* (1978), 263-265.
[13] ROBINSON, R. M. On the decomposition of spheres. *Fund. Math. 34* (1947), 226-260.
[14] SERRE, J.-P. *Corps locaux.* Act. Sci. Ind. 1296, Hermann éd., Paris 1962.
[15] DE SIEBENTHAL, J. Sur les sous-groupes fermés connexes d'un groupe de Lie clos. *Comm. Math. Helv. 25* (1951), 210-256.
[16] STEINBERG, R. Regular elements of semi-simple algebraic groups. *Publ. Math. I.H.E.S. 25* (1965), 49-80.
[17] TITS, J. Free subgroups in linear groups. *Journal of Algebra 20* (1972), 250-270.

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