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§7. THE MIDDLE RANGE

We return, as at the beginning of §4, to the study of intersections of curves in $\mathbf{C}^2$ with round disks D_r^4 and their boundaries S_r^3, and bidisks $D(r_1, r_2)$ and their boundaries. Now the (bi)radii are no longer required to be very small.

An embedding $i : (S, \partial S) \to (D_r^4, S_r^3)$ of a surface-with-boundary S into a round disk is a *ribbon embedding* provided that $N \circ i$ is a Morse function without local maxima on Int S, where $N(z, w) = |z|^2 + |w|^2$; and a surface-with-boundary $S \subset D_r^4$, with $\partial S = S_r^3 \cap S$, is a *ribbon surface* if the inclusion $(S, \partial S) \subset (D_r^4, S_r^3)$ is isotopic through embeddings of pairs to a ribbon embedding. To demand that a surface be ribbon is to place genuine topological restrictions on the embedding.

A theorem of Milnor [Mi 1], specialized to our dimensions, shows that if $\Gamma \subset \mathbf{C}^2$ is a nonsingular analytic curve then for almost all choices of origin and radius, the inclusion of $(\Gamma \cap D_r^4, \Gamma \cap S_r^3)$ into (D_r^4, S_r^3) is a ribbon embedding. A continuity argument easily shows that for no matter what choice of origin, $N \mid \Gamma$ has critical points, possibly degenerate, of index no greater than 1. It is easy to see that if Γ has singularities, an analogous theorem holds for $N \circ r : R \to [0, \infty[$ on the resolution. All these results generalize the Maximum Modulus Principle. Nothing much more seems to be known about big round disks and complex plane curves.

Turning our attention to bidisks, we let the way that they separate the variables z and w suggest an attitude to adopt towards our curves: consider one variable (conventionally w) as an analytic but possibly multiple-valued function of the other.

More precisely, let E_n be the space of unordered n-tuples of points of $\mathbf{C}$ (duplications allowed). Then E_n inherits a topology, and a structure of algebraic variety (affine, and singular if $n \geqq 2$), from its description as $\mathbf{C}^n/\mathscr{S}_n$, where the symmetric group $\mathscr{S}_n$ acts by permuting coordinates. Let E_n keep its topology, but normalize and resolve its algebraic variety structure, by using the map $\mathbf{C}^n \to E_n$ which carries $(c_1, ..., c_n)$ to $\{r_1, ..., r_n\}$ such that $(w - r_1) ... (w - r_n) = w^n + c_1 w^{n-1} + ... + c_n$. Now any function $F : X \to E_n$ can be called *an n-valued (complex) function on X*. The *graph* of an n-valued function on X is the obvious subset of $X \times \mathbf{C}$; adjectives like "continuous", "analytic", "algebraic" apply to n-valued functions in the obvious way.

We make the convention that (if X is not discrete) the entire image $F(X)$ should not lie in the subset $\Delta \subset E_n$ of unordered n-tuples with at least one duplication; Δ is an algebraic hypersurface (irreducible, and singular if $n \geqq 3$) in the affine space E_n, called the *discriminant locus.* Its complement $E_n - \Delta$ is called the *configuration space (of n distinct points in* **C***).*

To allow infinity as a value, we could replace **C** by $\mathbf{CP}^1$, E_n by $\mathbf{CP}^n$, and so on.

Let $f(z, w) \equiv f_0(z)w^n + f_1(z)w^{n-1} + \ldots + f_n(z) \in \mathbf{C}[z, w]$. Historically [Bl], the equation $f(z, w) = 0$ (or equivalently the curve it defines) was said to give w as an *algebraic function* of z, provided only that $f(z, w)$ was without repeated factors and without factors of the form $z - c$. (Also, of course, $f_0(z) \not\equiv 0$.) Then, in fact, on the complement in **C** of the zero-locus of $f_0(z)$, the assignment $z \mapsto \{w : f(z, w) = 0\}$ is an algebraic n-valued complex function. A zero of $f_0(z)$ is called a *pole* of the algebraic function, and can be accounted for by letting infinity be a value.

If $f_0(z) \equiv 1$, so that there are no poles at all, the algebraic function is *entire.* More generally, if $f_0(z), \ldots, f_n(z)$ are allowed to be entire functions of z (in the usual sense), then $f(z, w) = 0$ gives w as an n-valued *meromorphic function*; and if also $f_0(z) \equiv 1$, w is an *entire analytic n-valued function.* The graph of an n-valued entire function is a curve (algebraic or analytic as the case may be); when there are poles, the graph must be closed up to provide fibres over them.

Conversely, any algebraic curve in $\mathbf{C}^2$ becomes, after almost any linear change of coordinates, such a graph for some n. (This is not so for analytic curves, in general.) Thus we can study plane curves by studying certain curves in E_n.

Let $\gamma \subset \mathbf{C}$ be a simple closed curve, R the compact simply-connected region it bounds, $F : R \to E_n$ a continuous n-valued function analytic on Int R with $F(\gamma) \cap \Delta = \emptyset$. Then there is some radius $M > 0$ so that the graph of $F \mid \gamma$ lies in the open solid torus $\gamma \times \{w \in \mathbf{C} : |w| < M\}$; and this graph is a (not necessarily connected) n-sheeted covering space of γ. An application of one version of the Maximum Modulus Principle [G-R] shows that actually the graph of F itself is contained in $D = R \times \{w : |w| \leqq M\} \subset \mathbf{C}^2$, a topological 4-ball (with boundary 3-sphere piecewise as smooth as γ). Now, $F(R) \cap \Delta$ must be finite; let $F^{-1}(\Delta) \subset R$ be called the *branch locus*, and denoted B. One can easily see that the graph of F in D is a 2-dimensional pseudomanifold-with-boundary (i.e., geometric relative cycle), with any singularities lying in $B \times \{w : |w| < M\} \subset \mathrm{Int}\, D$; its boundary in ∂D is exactly the link L which is the graph of $F \mid \gamma$. Furthermore, the graph of F is naturally oriented (by its complex structure at the regular points), so L has a natural orientation, and the projection $L \to \gamma$ preserves orientations.

At this point it is convenient to introduce braids; a general reference is [Bi]. The *braid group on n strings* is the fundamental group $B_n = \pi_1(E_n - \Delta; *)$ of the configuration space. Let $l : [0, 2\pi] \to E_n - \Delta$, $l(0) = l(2\pi)$, be a parametrization of a loop in the configuration space. Then the graph of l in

$[0, 2\pi] \times \mathbf{C}$ is a *geometric braid*, that is, the union of disjoint arcs, on which pr_1 is a covering projection to $[0, 2\pi]$, and such that the unordered n-tuples of top and bottom endpoints are identical; each arc is called a *string*. Under the map $[0, 2\pi] \times \mathbf{C} \to S^1 \times \mathbf{C} : (\theta, w) \mapsto (e^{i\theta}, w)$, a geometric braid is carried to a *closed braid* in the open solid torus. When $S^1 \times \mathbf{C}$ is identified with the tubular neighborhood of an unknotted circle in S^3, in such a way that distinct circles $S^1 \times \{z_0\}$ and $S^1 \times \{z_1\}$ are (algebraically, and therefore geometrically) unlinked, then any closed braid becomes a knot or link in S^3, and it is naturally oriented. For $\beta \in B_n$, any closed braid constructed in this way from a loop which represents β is denoted $\hat{\beta}$. If, conversely, $L \subset S^1 \times \mathbf{C}$ is an oriented link on which pr_1 is an orientation-preserving n-sheeted covering map, then any choice of a basepoint $e^{i\theta} \in S^1$ yields a loop in $E_n - \Delta$, based at $* = \{w \in \mathbf{C} : (e^{i\theta}, w) \in L\}$, and thus a braid $\check{L} \in B_n = \pi_1(E_n - \Delta; *)$, with $(\check{L})^\wedge = L$.

Since Δ is irreducible, the abelianization of B_n is infinite cyclic, and in fact B_n is normally generated by one element, that is, generated by a single conjugacy class. Choose for the basepoint $*$ of $E_n - \Delta$ the (real) n-tuple $\{1, ..., n\}$. Let

$$g_i(z, w) \equiv \left(w^2 - (2i+1)w + \left(i^2 + i + \frac{1}{4}(1-z)\right)\right) \cdot \prod_{\substack{j=1 \\ j \neq i, i+1}}^{n-1} (w-j) \in \mathbf{C}[z, w],$$

for $i = 1, ..., n-1$; and let $G_i : \mathbf{C} \to E_n$ be the n-valued function corresponding to $g_i(z, w) = 0$. If $R = \{z : |z| \leqq 1\}$, then each $G_i \mid R$ is an embedding of R as a normal disk to Δ (at a regular point), with center

$$G_i(0) = \left\{1, ..., i-1, i + \frac{1}{2}, i + \frac{1}{2}, i+2, ..., n\right\}$$

on Δ, and basepoint $G_i(1) = *$. Giving ∂R its positive (counterclockwise) orientation, we get oriented loops in $E_n - \Delta$, and the homotopy class of $G_i(\partial R)$ is denoted by σ_i and called *the i-th standard generator of* B_n. (The geometric braids corresponding to the given construction are the standard pictures of the σ_i.) The set of standard generators does, in fact, generate B_n, cf. [Bi]. Each σ_i is conjugate to σ_i. Following [Ru 2], let any braid in B_n conjugate to σ_1 be called a *positive band in* B_n; a loop in the configuration space represents a positive band if and only if it is the oriented boundary of an oriented disk in E_n which meets the discriminant locus transversely in a single positive (regular) point. The inverse of a positive band is a *negative band*.

An ordered k-tuple $\mathbf{b} = (b(1), ..., b(k))$ of bands in B_n is a *band representation of length* k *of the braid* $\beta(\mathbf{b}) = b(1) \cdots b(k)$. (A braid word is a band representation where each band is a standard generator or the inverse of a standard generator.) Each braid has many band representations, corresponding to the various null-homotopies, transverse to Δ, of a loop representing the braid in $E_n - \Delta$ to a point in E_n. (See [Ru 2] for a precise statement and proof.) Such a null-homotopy gives a map of a disk into E_n, transverse to Δ—the length of any corresponding band representation is the geometric number of intersections of the disk with Δ, and the number of positive (resp., negative) bands is the number of positive (resp., negative) intersections with Δ. In particular, suppose each such intersection is positive, so each band $b(s)$ is positive. Then $\mathbf{b}$, $\beta(\mathbf{b})$, and the closed braid $\hat{\beta}(\mathbf{b})$ are all called (in [Ru 1-4]) *quasipositive*. The closed braid L, associated to an analytic n-valued function F and a simple closed curve γ which bounds a simply-connected region in the domain of F, is quasipositive. (If F as given is not transverse to Δ in R, almost any small translation of F in E_n will become so, while the braid type of L won't change; and complex analytic intersections are positive.)

Conversely, it is shown in [Ru 1] that for every quasipositive band representation $\mathbf{b}$ in B_n, there are an algebraic n-valued function and simple closed curve yielding the given band representation in the manner just exposed. It is also shown (and this is why we have excluded poles) that any type of closed braid whatever can occur as the graph over S^1 of a meromorphic (algebraic) n-valued function on $\mathbf{C}$. (But note that when poles actually do occur inside the simple closed curve, the closed braid is never the complete boundary of the piece of analytic curve inside a bidisk; a typical example is given by $f(z, w) \equiv zw - \frac{1}{4}$, in $D(1, 1)$, $\gamma = S^1$.)

Let $e : B_n \to \mathbf{Z}$ be abelianization. Thus $e(\beta)$ is the *exponent sum* of β, when β is written as a braid word in the standard generators; or more generally it is the number of positive bands in $\mathbf{b}$, minus the number of negative bands in $\mathbf{b}$, when $\beta(\mathbf{b}) = \beta$. Geometrically, $e(\beta)$ is the linking number of (any loop representing) β with Δ, in E_n. Analytically, $e(\beta)$ can be given by an integral formula, as by Laufer [Lau], where it is called *self-winding* (and is generalized to links that aren't necessarily given as closed braids).

When $\mathbf{b}$ is quasipositive, $e(\beta(\mathbf{b}))$ is the length of $\mathbf{b}$, a fact with the following geometric meaning. When $F : R \to E_n$ is smooth and transverse to Δ, then the graph of F is a smooth surface in $R \times \mathbf{C}$; the intersections with Δ correspond to "simple vertical tangents" to the graph, and projection from the graph of F back to R is a branched covering, with only two sheets coming together over each branch point in R. Thus the Euler characteristic $\chi(\text{graph } F)$ equals $n\chi(R) - l$, if l is the number of branch points. When R is a disk and F corresponds to a quasipositive band representation $\mathbf{b}$, then l is the length of $\mathbf{b}$ and we recover a genus formula for the graph of F in terms of n, the number of components of the boundary of the graph, and the exponent sum of the boundary. More generally, when F is analytic, even if it is not transverse to Δ it will have a well-defined positive intersection multiplicity at each point of intersection, which will equal the number of geometric intersections of almost any small (analytic) perturbation of F; thus its graph, which will now be a singular curve, will have well-defined multiplicities for each singular point, and again a genus formula can be recovered, this time involving also these multiplicities: cf. [Lau].

A very interesting subclass of the quasipositive braids consists of the *positive braids*. A braid in B_n is positive if it can be written as a word in the standard generators without using their inverses, *strictly* positive if each of $\sigma_1, \ldots, \sigma_{n-1}$ actually occurs. Positive braids play an important algebraic role in the braid group (cf. [Bi]). Closed positive braids enjoy various nice knot-theoretical properties (cf. [St], [Ru 5]), and have turned up in diverse contexts—as knotted orbits of some special dynamical systems [Bi-W]; and, what is relevant here, as the links of singular points of plane curves.

Let $f(z, w) \in \mathbf{C}[z, w]$ be squarefree, not divisible by z, and satisfy $f(0, 0) = 0$. Then for $\varepsilon > 0$ sufficiently small, $f(z, w) = 0$ defines an n-valued analytic function $F : \{z : |z| \leqslant \varepsilon\} \to E_n$ with $F^{-1}(\Delta) = \{0\}$. Let $w_1(z), \ldots, w_n(z)$ be the n numbers in $F(z)$; then it is readily seen that the assignment $z \mapsto \{w_i(z) - w_j(z) : 1 \leqslant i, j \leqslant n, i \neq j\}$ is an $n(n-1)$-valued analytic function. Without loss of generality, we may take n and ε so that $w_1(0) = \ldots = w_n(0) = 0$, and $w_i(z) - w_j(z) \neq 0$ for $z \neq 0, |z| \leqslant \varepsilon$. Now a straightforward calculation shows that for $z \neq 0, |z| \leqslant \varepsilon$, we have $d(\arg(w_i - w_j))/d(\arg z) > 0$. Consider the closed braid L which is the graph of $F \mid \{z : |z| = \varepsilon\}$, and the link of the singularity of $\{f = 0\}$ at $(0, 0)$. A braid diagram for L may be obtained by projecting its ambient solid torus $S^1 \times \mathbf{C}$ onto $S^1 \times e^{i\theta}\mathbf{R}$ orthogonally; for almost all θ this will be a braid diagram in general position, from which a braid word may be read off in the usual way; and the signs of the crossings are precisely determined as the signs at the appropriate points of $d(\arg(w_i - w_j))/d\theta$. Since $\theta = \arg z$, the link of a singularity is a positive closed braid. In fact, it can be seen to be strictly positive; for if it were not, it would be a split link, in particular it would have components with zero algebraic linking—but the linking number of two components of the link of a singularity is the intersection number of the corresponding branches, and is strictly positive.

It is known that a strictly positive closed braid is a fibred link, cf. [St], [Bi-W], which provides another proof (in this dimension) of Milnor's Fibration Theorem (that the link of a singularity is fibred—Milnor, of course, gives an actual analytic formula for the fibration). Here is a simple proof which

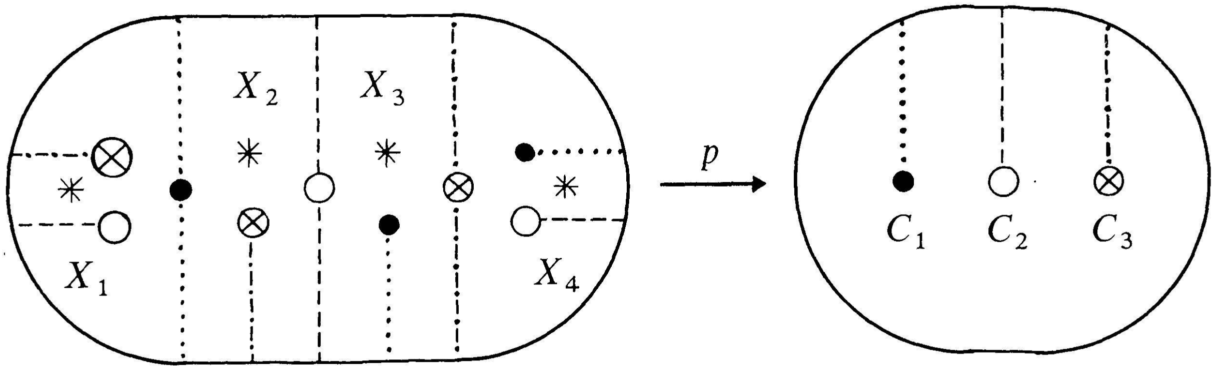

FIGURE 1 ($n=4$)

geometrically constructs a fibration of the complement of a strictly positive closed braid. Let $p : X \to \mathbf{C}$ be the n-sheeted branched covering with branch locus $\{1, ..., n-1\}$, where the permutation at j is the transposition $(j \quad j+1)$. Then X is homeomorphic to $\mathbf{C}$ again. For concreteness, we realize p as in Figure 1: cuts $C_j = \{w : \operatorname{Re} w = j, \operatorname{Im} w \geqq 0\}$ are made in the base space; we coordinate X so that the singular point of $p^{-1}(j)$ is j, and so that $\{z : \operatorname{Re} z = j\}$ is one component of $p^{-1}(C_j)$; then the components of $p^{-1}(\mathbf{C} - \bigcup_{j=1}^{n-1} C_j)$ are the sets

$$X_1 = \{z : \operatorname{Re} z < 1\}, X_2 = \{z : 1 < \operatorname{Re} z < 2\}, ..., X_n = \{z : n-1 < \operatorname{Re} z\},$$

known in the classical style as *sheets* of the branched cover. Now if we consider $E_n - \Delta$ to be the configuration space of X, the inverse of the covering map defines a continuous function from $\mathbf{C} - \{1, ..., n-1\}$ into $E_n - \Delta$, inducing a homomorphism from the free group $\pi_1(\mathbf{C} - \{1, ..., n-1\}; 0)$ to the braid group $\pi_1(E_n - \Delta; p^{-1}(0))$. One readily checks that this homomorphism is onto, carrying the obvious free generator x_j of the free group (Figure 2) to the standard generator $\sigma_j \in B_n$. Let $v = x_{j(1)} \dots x_{j(k)}$ be any strictly positive word in

$$x_1, ..., x_{n-1}, \beta = \sigma_{j(1)} \cdots \sigma_{j(k)} = (p^{-1})_*(v)$$

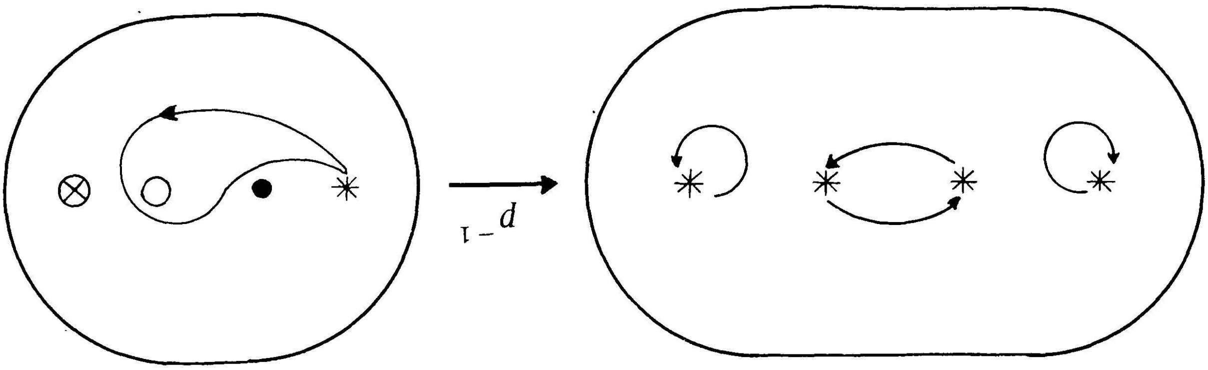

FIGURE 2 ($n=4$)

$(p^{-1})_*(x_2) = \sigma_2$

its strictly positive image in B_n. We use v to construct an auxiliary closed braid in $S^1 \times \mathbf{C}$, the closure of $v' = A_{1,j(1)} \dots A_{1,j(k)} \in B_{n+1}$, where $A_{1,j} = (\sigma_1 \dots \sigma_{j-1})\sigma_j^2(\sigma_1 \dots \sigma_{j-1})^{-1}$ is one of the standard generators $A_{i,j}$ of the *pure braid group* (cf. [Bi] or see below). Now, v' can be realized as a geometric braid in two special ways: the first string can be made to wind in and out among the others, which are all straight; or the first string may be made straight, while the others wind around it in a succession of loops (Figure 3). On the first

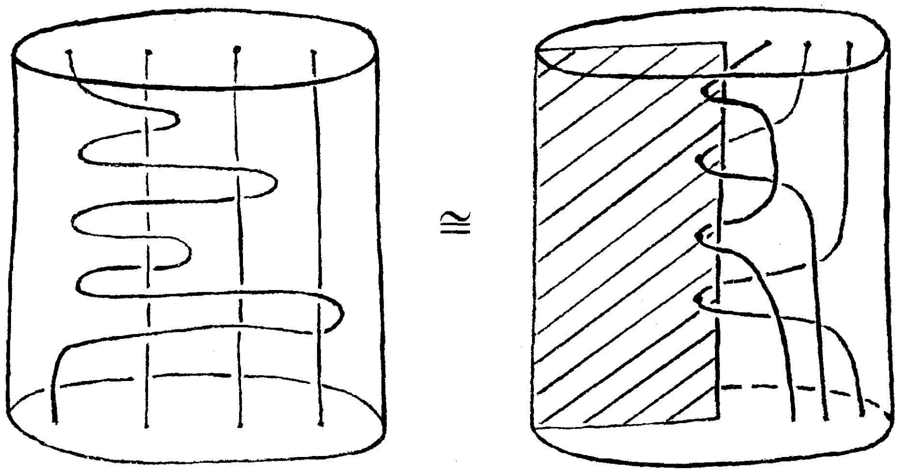

FIGURE 3 ($n=4$)
$v = x_1 x_2 x_1 x_3$

interpretation, identifying the straight strings with $[0, 2\pi] \times \{1, \dots, n-1\}$, the winding first string becomes the graph of a loop

$$l : ([0, 2\pi], \{0, 2\pi\}) \to (\mathbf{C} - \{1, \dots, n-1\}, 0\})$$

in the homotopy class v; and its inverse image under the branched covering $id_{S^1} \times p : S^1 \times X \to S^1 \times \mathbf{C}$ is a geometric braid representing β. On the second interpretation, identifying the single straight string with $[0, 2\pi] \times \{0\}$, and taking care that each other string winds monotonically around this axis, the fibration of $S^1 \times (\mathbf{C} - \{0\})$ over S^1 by $(e^{i\theta}, w) \mapsto \arg w$ lifts back through the branched covering to a fibration of $(S^1 \times X) - \hat{\beta}$ over S^1. (The strictness is used at this point, to ensure that in fact there is a non-zero winding number for each string. Positivity, however, could be weakened to "homogeneity" in the sense of [St].) There is no trouble "at infinity", so that the fibration can be extended over all of $S^3 - \hat{\beta}$. Note that the fibre surface for $\hat{\beta}$ is the union of n disks with a surface that is the cover of an annulus branched at $e(\beta)$ points, so it has Euler characteristic $n - e(\beta)$ and hence (being connected) genus $g = 1 - \frac{1}{2}(n - e(\beta) + c)$ if $\hat{\beta}$ has c components. This is the same genus formula as before when the link of a singularity is considered.

Besides exponent sum, there are other representations of B_n with applications here. First recall the *permutation representation* $\pi : B_n \to \mathscr{S}_n$, which takes σ_j to $(j \;\; j+1)$, $j = 1, \ldots, n-1$. The kernel $\ker \pi$ is the group of *pure braids*; it is the fundamental group of the space of *ordered* n-tuples of distinct complex numbers. Let S_n be the free abelian group of rank $\frac{1}{2}n(n-1)$ consisting of symmetric n-by-n integer matrices with 0 diagonal. Now, in general, a cycle in $\pi(\beta)$ corresponds to a component of $\hat{\beta}$; and in particular the closure of a pure braid consists of n (unknotted) components which are naturally ordered $1, \ldots, n$. Define $\lambda : \ker \pi \to S_n$ by setting $\lambda(\beta)_{i,j}$ equal to twice the linking number of the i-th and j-th components of $\hat{\beta}$, for β pure. These representations are combined in $\omega : B_n \to S_n \ltimes \mathscr{S}_n$, where in the semidirect product $\mathscr{S}_n$ acts on S_n by conjugation with the standard permutation matrices, and

$$\omega(\sigma_i) = ([\delta_{i,i+1} + \delta_{i+1,i}], (i \;\; i+1)), i = 1, \ldots, n-1 .$$

Let $\mathscr{S}_n$ act diagonally on $\{1, \ldots, n\}^2$, and let $|x| \cdot (i, j)$ denote the orbit of (the cyclic subgroup generated by) $x \in \mathscr{S}_n$ on (i, j). Then for

$$i \neq j, \beta \in B_n, \omega(\beta) = ([a_{pq}], x),$$

the sum $\sum_{(p,q) \in |x| \cdot (i,j)} a_{pq}$ is an integer invariant of β, and appropriate sums of such invariants are conjugacy class invariants. In particular, when $\pi(\beta)$ is an n-cycle (so that $\hat{\beta}$ is a knot), such a conjugacy class invariant arises by summing over pairs (i, j) with a fixed constant difference modulo n: and this may be seen to be precisely twice one of the *self-windings* sw_i introduced by Laufer [Lau]. Laufer showed that the $sw_i (i = 1, \ldots, n)$ suffice to distinguish the knot types of links of unibranch singularities; in fact, he showed that the Puiseux pairs of a branch could be reconstructed from the self-windings. Simple examples show that $sw = e$ and the sw_i (and even their slight generalizations just given) can't tell apart all quasipositive, or even all positive, closed braids. It is interesting to speculate that there might be reasonable representations λ_1 of $\ker \lambda$, λ_2 of $\ker \lambda_1, \ldots$, which could somehow be combined into a (faithful?) representation of B_n in which quasipositivity might show up more clearly than it does in B_n itself. (Is there any relation to Laufer's other numerical link invariants [Lau 2]? Perhaps λ_1 can be constructed out of linking numbers in branched covers of S^3, branched over one of the—unknotted!— components of a pure link in which every linking number is 0; and so on.)

As a final topic, we return to "knot groups" of plane curves and related matters, from a braid-theoretical point of view.

As before, let R be the compact region of $\mathbf{C}$ bounded by a simple closed curve γ. Let S be a compact oriented surface-with-boundary. Then a map $f : S \to R \times \mathbf{C}$, or its image $f(S)$, is a *braided surface* of degree $n \geqq 1$ provided that $pr_1 \circ f : S \to R$ is a branched covering of degree n: f is a *smooth, analytic,* or *algebraic* braided surface if $f(S)$ is smooth, complex analytic, or (complex) algebraic. Let $V'_f \subset S$ and $V_f \subset R$ denote the branch sets of $pr_1 \circ f$, finite sets avoiding ∂S and γ; and let W_f, $V_f \subset W_f \subset R$, be the set $\{z \in R : (\{z\} \times \mathbf{C}) \cap f(S)$ contains fewer than n points$\}$. One can interpret f^{-1} as a map, as smooth as f, from R into E_n. As remarked earlier, when f^{-1} is transverse to Δ, then $W_f = V_f$ and f is a smooth braided surface; but f can be smooth without f^{-1} being transverse to Δ. (Consider non-generic "vertical" tangencies.) Nor need W_f be finite, *but we will always assume that it is*, even when f^{-1} is not transverse to Δ. With this proviso, *every braided surface* f is a topological (even p.l.) immersion, though not necessarily locally flat. To see this, define the *local braid of* f *at* $z \in R$, denoted $\beta_{f,z} \in B_n$, to be the homotopy class of the loop $\theta \to f^{-1}(z+\varepsilon e^{i\theta})$, $0 \leqq \theta \leqq 2\pi$, for any sufficiently small $\varepsilon > 0$. (Since the basepoints of the various copies of B_n vary with z, $\beta_{f,z}$ is really only defined up to conjugacy.) This is well-defined when W_f is finite (or even as long as z is not an accumulation point of W_f); of course $\beta_{f,z} = 1$ if and only if $z \in R - W_f$. For $z \in W_f$, $\hat{\beta}_{f,z}$ has strictly fewer than n components, which will be grouped into possibly yet fewer unsplittable links. Then $f(S)$, above z, is embedded in $R \times \mathbf{C}$ like disjoint cones (with distinct vertices) on the unsplittable sublinks of $\hat{\beta}_{f,z}$. For example, if $z \in V_f$ lies under only a simple vertical tangent, then $\beta_{f,z}$ is a band (positive or negative), which might as well be taken to be $\sigma_1^{\pm 1} \in B_n$, and $\hat{\beta}_{f,z}$ is a split link of $n-1$ unknotted components.

Recall (cf. [Bi]) that B_n acts (faithfully) as a group of automorphisms of the free group F_n of rank n. Explicitly, if $F_n = \pi_1(\mathbf{C} - \{w_1, \ldots, w_n\}; w_0)$, the acting B_n is realized as $\pi_1(E_n - \Delta; \{w_1, \ldots, w_n\})$; on standard free generators $x_1, \ldots, x_n$ of F_n (positively oriented meridians around $w_1, \ldots, w_n$), the action is

$$x_i\sigma_i = x_i x_{i+1} x_i^{-1}, \; x_{i+1}\sigma_i = x_i, \; x_j\sigma_i = x_j$$

for $j \neq i, i+1$. Pick a basepoint $z_0 \in R - W_f$, and paths from z_0 to the points $z_1, \ldots, z_k$ of W_f. By these paths, all the local braids can be taken to lie in one and the same braid group, namely, $\pi_1(E_n - \Delta; pr_2((\{z_p\} \times \mathbf{C}) \cap f(S)))$—denote by $\beta'_{f,z}$ these braids. (Simple vertical tangents, for instance, will now give braids $\beta'_{f,z}$

which are bands that *cannot* all at once be taken to be $\sigma_1^{\pm 1}$.) It may now be seen that

$$(x_1, \ldots, x_n : x_i\beta'_{f,z} = x_i, i = 1, \ldots, n, z \in W_f)$$

is a presentation of the "knot group" $\pi_1((R \times \mathbf{C}) - f(S); *)$. When f is algebraic and γ is a very large circle this is really van Kampen's presentation (except for the relation "at infinity" to which we will return shortly).

A finite presentation of a group, in which each relation sets one generator equal to some conjugate of another generator, is a *Wirtinger presentation*; a group with a Wirtinger presentation is a *Wirtinger group*. Any Wirtinger group has a *simple* Wirtinger presentation, in which each relation is of the form $x_ix_jx_i^{-1} = x_k$, for not necessarily distinct generators x_i, x_j, x_k. After possibly adding more generators, and renumbering them, one can assume that each relation is of one of the two forms $x_i = x_{j+1}$ or $x_i = x_jx_{j+1}x_j^{-1}$, $i < j$. These two relations are contributed, respectively, by the action on F_n of

$$(\sigma_i\sigma_{i+1} \ldots \sigma_{j-1}^{\varepsilon})\sigma_j(\sigma_i\sigma_{i+1} \ldots \sigma_{j-1}^{\varepsilon})^{-1}, \varepsilon = +1 \text{ or } -1 .$$

So every Wirtinger group has a simple Wirtinger presentation which is the van Kampen presentation of the fundamental group $\pi_1(\{(z, w) \in \mathbf{C}^2 : |z| \leqq 1\} - f(S); *)$ for some smooth braided surface $f(S)$ with boundary the closure of a quasipositive braid (the product of the bands used to achieve the desired relations); and actually $f(S)$ can be taken to be non-singular complex analytic. So we see that *the class of knot groups of complex analytic curves in a bidisk is exactly the class of Wirtinger groups*, a refinement [Ru 2] of results of Yajima [Ya] and Johnson [Jo] (who weren't concerned with complex analytic structures).

If one wishes to investigate knot groups for smooth braided surfaces of fixed topological type, one still loses nothing by demanding that the surfaces be complex curves: if $f(S)$ is smooth, by slight jiggling f^{-1} becomes transverse to Δ while $f(S)$ moves by an isotopy; then the braids $\beta'_{f,z}$ are all bands, positive or negative; changing all the signs to positive reimbeds S as a quasipositive braided surface, and therefore, up to isotopy, a complex analytic curve; but it does not change the knot group at all, since $x\beta^{-1} = x$ is the same relation as $x = x\beta$.

So far, everything has been phrased for braided surfaces over a compact (simply-connected) region R. If we replace R by all of $\mathbf{C}$, much stays the same; it is now appropriate to let W_f be infinite, but discrete. It ceases to be clear, however, (at least to this author at the present time) that a quasipositive "infinite band representation" can always be realized by an entire n-valued analytic function. Also, as observed in [Ru 1], for compact R, at least as far as the boundary closed braid is concerned, every n-valued analytic function can be assumed to be the restriction of an entire n-valued *algebraic* function; this is certainly not true for $R = \mathbf{C}$, because the "local braid at infinity" $\beta_{f,\infty}$ of an algebraic braided surface over $\mathbf{C}$—i.e., the braid over a simple closed curve large enough to enclose V_f entirely—is severely restricted. Its closure, for instance, is an iterated torus link (as we saw in the proof of the theorem of Abhyankar and Moh, § 6). And if the projective completion of the algebraic braided surface (algebraic curve), in $\mathbf{CP}^2$, meets the line at infinitely transversely, one actually has $\hat{\beta}_{f,\infty}$ the union of n circles of the Hopf fibration $S^3 \to \mathbf{CP}^1$—the braid $\beta_{f,\infty}$ is the generator of the (infinite cyclic) center of $B_n(n \geqslant 3)$, which bears the name Δ^2 (unfortunately, in this context), cf. [Bi]. Any knot group of a projective plane curve, then, can be presented by starting with an expression of Δ^2 as a product $\beta(l) \cdots \beta(k)$ in B_n, where each $\beta(i)$ is conjugate in B_n to some local braid associated to the link of a singularity (including non-trivial local braids which are associated to the unknotted link of a regular point!), then forming the presentation

$$(x_1, \ldots, x_n \colon x_1 x_2 \cdots x_n = 1,\ x_i\beta(j) = x_i,\ i = 1, \ldots, n,\ j = 1, \ldots, k)\,.$$

For instance, a quasipositive band representation of Δ^2 (each $\beta(i)$ a positive band, that is, conjugate to the nontrivial local braid σ_1 associated to a simple vertical tangent) corresponds to a non-singular curve of degree n, and presents $\mathbf{Z}/n\mathbf{Z}$. A quasipositive *nodal* band representation, where each $\beta(i)$ is either a positive band or the square of a positive band, corresponds to a node curve; if some $\beta(i)$ are cubes of positive bands, others squares or first powers, we have a *cuspidal* band representation; and so on. There is a mapping from the set of strata of Q_n (§ 5) into a hierarchy of "types of expressions" of $\Delta^2 \in B_n$ as products $\beta(l) \cdots \beta(k)$; Moishezon's problem of normal forms is a first step in the study of this mapping, about which little seems to be known. Is it onto? An affirmative answer would be a strong generalization of Riemann's Existence Theorem. (Again, cf. [Mo].)

We conclude with three examples. First recall some formulas for Δ^2 in B_n (cf. [Bi] or [Mo]): $\Delta^2 = (\sigma_1\sigma_2\cdots\sigma_{n-1})^n$; also, Δ^2 is pure, and in terms of the standard generators

$$A_{ij} = (\sigma_i \cdots \sigma_{j-1})\, \sigma_j^2 (\sigma_i \cdots \sigma_{j-1})^{-1},\ 1 \leqslant i \leqslant j \leqslant n-1\,,$$

of the pure braid group,

$$\Delta^2 = A_{1,n-1}A_{1,n-2} \cdots A_{1,1}A_{2,n-1} \cdots A_{2,2} \cdots A_{n-1,n-1}\,.$$

Example 1. Write $\Delta^2 = \beta(l) \cdots \beta(n^2 - n)$, $\beta(i) = \beta_{i \bmod n-1}$, as just given. It is easy to see that this expression for Δ^2 does in fact correspond to a non-singular curve of degree n. The corresponding presentation of the knot group of the curve includes among its relations $x_1 x_2 \cdots x_n \doteq 1$ and each equality $x_i = x_{i+1}$, $i = 1, \ldots, n-1$. So the knot group is a quotient of $\mathbf{Z}/n\mathbf{Z}$; but a simple homological argument shows that $\mathbf{Z}/n\mathbf{Z}$ is the abelianization of the knot group, so the two groups are equal.

Example 2. Write $\Delta^2 = \beta(l) \cdots \beta((n^2-n)/2)$, where $\beta(i) = A_{p,q}$ as above. Each pair (p, q) arises. The relations in the corresponding presentation say that for each pair p, q the generators x_p, x_{q+1} commute. (For instance, the action of

$$A_{1,1} = \sigma_1^2 \text{ on } F_n \text{ is } x_1\sigma_1^2 = (x_1 x_2 x_1^{-1})\sigma_1 = x_1 x_2 x_1 x_2^{-1} x_1^{-1},$$
$$x_2\sigma_1^2 = x_1\sigma_1 = x_1 x_2 x_1^{-1},\ x_k\sigma_1 = x_k,\ k \neq 1, 2;$$

and the relations $x_1 = x_1 x_2 x_1 x_2^{-1} x_1^{-1}$ and $x_2 = x_1 x_2 x_1^{-1}$ both say x_1 commutes with x_2.) The group is free abelian of rank $n-1$. Moishezon sketches a proof that this presentation does arise geometrically; another proof could be given by the method of [Ru 1].

Example 3. For $n = 4$, $\Delta^2 = \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_3$. Let us suppress the symbol σ, raise subscripts (so k denotes σ_k), and write, for instance, ${}^{2\bar{3}}4$ to mean $\sigma_2\sigma_3^{-1}\sigma_4\sigma_3\sigma_2^{-1}$. Then, by dogged manipulation, $\Delta^2 \in B_4$ can be worked into the form $(3\cdot3\cdot3)$ $({}^{3\bar{2}}1)$ $(1\cdot1\cdot1)$ (2) $(1\cdot1\cdot1)$ $({}^{32}1)$, the product of three positive bands and three "cusps" (cubes of positive bands). The corresponding presentation, before adjoining the relation at infinity, presents the group of the 5-twist spun trefoil (as has been remarked by Dewitt Sumners); with that relation, $x_1x_2x_3x_4 = 1$, the group becomes the non-abelian group of order 12, $(a, b : aba = bab,\ a^4 = 1,\ a^2 = b^2)$. This is the correct group [Z] for a tricuspidal cubic curve, and presumably the given "quasipositive cuspidal band representation" really arises geometrically, but I have not had the courage to check this.—Similarly, for $n = 6$, $\Delta = 123451234123121$, which can be written as $(1\cdot1\cdot1)$ $({}^{\bar{1}2}1)$ $(3\cdot3\cdot3)$ $({}^{\bar{3}4}3)$ $(5\cdot5\cdot5)$ $({}^{\overline{133}}2)$ $({}^{\overline{355}}4)$ $({}^{2}3)$ $({}^{4}5)$; the presentation for the square of this, with the relation at infinity, is at an intermediate stage $(x_1, x_2, x_3, x_4, x_5, x_6 : x_1 = x_3 = x_5,\ x_2 = x_4 = x_6,\ x_1x_2x_1 = x_2x_1x_2,\ x_1x_2x_3x_4x_5x_6 = 1)$ which becomes $(a, b:\ a^2 = b^3 = 1)$, the group given in [Z] for a sextic with six cusps on a conic. On the other hand, a less symmetrical way to write $\Delta^2 \in B_6$ is as $({}^{2\bar{1}2}3)$ (4) (5) $(2\cdot2\cdot2)^1$ $(2\cdot2\cdot2)$ $({}^{3}2)$ $({}^{431}2)$ $(1\cdot1\cdot1)$ $({}^{4\bar{3}2}1)$ $({}^{44}5)$ $\cdot$ $({}^{442}3)$ $(4\cdot4\cdot4)$ $(1\cdot1\cdot1)$ $({}^{2\bar{1}2}3)^2$ $(1\cdot1\cdot1)$ $({}^{22}1)$ (2), which presents the abelian group $\mathbf{Z}/6\mathbf{Z}$ which [Z] gives for a sextic with six cusps *not* all on the same conic.