§0. Introduction

with boundary (possibly empty). Since all closed P.L. manifolds of dimension less than 8 have smooth structures they are homeomorphic to algebraic sets. In 1968 Kuiper [Ku] extended Nash's result to all 8 dimensional closed P.L. manifolds. Later in [A] it was shown that all 8-dimensional closed P.L. manifolds as well as a larger class of nonsmoothable polyhedra are homeomorphic to real algebraic sets. All these results use transversality and local piecing techniques which in general does not work when dealing with singular spaces. In [AK_1], [AK_2], [AK_5] a resolution technique was introduced. Namely, by constructing a "topological" resolution of a singular space one gets a smooth manifold, then by isotoping this to a nonsingular algebraic set and algebraically blowing it down, one puts an algebraic structure on the original singular space. Using this in [AK_2] a complete topological characterization for algebraic sets with isolated singularities was given. Later it was established that the interior of all compact P.L. manifolds are P.L. homeomorphic to real algebraic sets; in fact these algebraic structures are classified up to topological concordances [AK_6], [AT_2].

In this paper we give an overview of these results. For the sake of harmony we sketch proofs when possible. We have reproduced some of [AK_7] since it has not appeared in print. The last section (§6) is a summary of our ongoing work; so it is somewhat tentative. We hope to give a more complete and final account in [AK_9]. The first named author would like to thank C. Weber and M. Kervaire for their hospitality during this conference in Switzerland.

§0. Introduction

A *real algebraic set* V is a set of the form

$$V(I) = \quad x \in R^n \mid p(x) = 0, p \in I\}$$

where I is a set of polynomial functions from $\mathbf{R}^n$ to $\mathbf{R}$. We can write any algebraic set $V = p^{-1}(0)$ where $p(x)$ is a single polynomial (p is the sum of the square of the generators of I). $V(J)$ is called an *algebraic subset* of $V(I)$ if $I \subset J$. An algebraic set V is called *irreducible* if it can not be written as a union of two algebraic sets $V_1 \cup V_2$ with each $V_i \neq V$. If V is an algebraic set then $I(V)$ denotes the ideal of polynomials vanishing on V. A point $x \in V$ is called *nonsingular of dimension* d if there is a polynomial function $p: \mathbf{R}^n \to \mathbf{R}^{n-d}$ vanishing on V and an open neighborhood U of x with the property that rank$(dp) = n - d$ on U and $p^{-1}(0) \cap U = V \cap U$. dim$(V)$ is defined to be the largest d such that there is a $x \in V$ of nonsingular of dimension d. Nonsing(V) is the set of all $x \in V$ which are nonsingular of dimension $\dim(V)$. Then we define $\mathrm{Sing}(V) = V - \mathrm{Nonsing}(V)$. An interesting fact is that if W and V are nonsingular algebraic sets of the same dimensions with $W \subset V$ then $V - W$ is a nonsingular algebraic set (Lemma 1.6 of [AK_2]).

For any set $A \subset \mathbf{R}^n$ the Zariski closure $\bar{A}$ of A is defined to be the smallest algebraic set containing A. Given algebraic sets $V \subset \mathbf{R}^n$ and $W \subset \mathbf{R}^m$ a function $f: V \to W$ is called an *entire rational function* if $f(x) = p(x)/q(x)$ where $p: \mathbf{R}^n \to \mathbf{R}^m$, $q: \mathbf{R}^n \to \mathbf{R}$ are polynomials such that q does not vanish on V. A diffeomorphism $f: V \to W$ is called a *birational diffeomorphism* if f and f^{-1} are entire rational functions.

Consider $E(n, k) \xrightarrow{p} G(n, k)$ where $G(n, k)$ is the Grassmann manifold of k-planes in $\mathbf{R}^n$ $E(n, k)$ is the universal bundle over $G(n, k)$. These universal manifolds are nonsingular algebraic sets in a natural way

$$G(n, k) = \{A \in \mathcal{M}_n \mid A = A^t, A^2 = A, \mathrm{trace}(A) = k\}$$
$$E(n, k) = \{(A, x) \in G(n, k) \times \mathbf{R}^n \mid Ax = x\}$$

where $\mathcal{M}_n$ is the space of $(n \times n)$ matrices $(= \mathbf{R}^{n^2})$ and $p(A, x) = A$. For a given pair of nonsingular algebraic sets $M \subset V \subset \mathbf{R}^n$ of dimensions m and v, the usual functions

$$f: M \to G(n, v-m), \qquad g: M \to G(n, m),$$

$f(x) =$ the $(v-m)$-plane tangent to V and normal to M at x, $g(x) =$ the m-plane tangent to M at x are entire rational functions (see [AK_2], [AK_3]). There is a birational diffeomorphism $\theta: \mathbf{RP}^{n-1} \to G(n, 1)$ given by $\theta[x_1; \ldots; x_n] = (a_{ij})$ where $a_{ij} = \dfrac{x_i x_j}{\Sigma x_i^2}$. Then $V \subset \mathbf{RP}^{n-1}$ is a projective algebraic set if and only if $\theta(V)$ is an algebraic subset of $G(n, 1) \subset \mathbf{R}^{n^2}$. Hence every projective algebraic set is an afine algebraic set and vice versa.

In real algebraic geometry locally defined entire rational functions are globally defined. This property does not hold in the complex case.

LEMMA 0.1. *Let $\{V_i\}_{i=1}^k$ be disjoint algebraic subsets of an algebraic set V, and $f_i: V_i \to \mathbf{R}^n$ be entire rational functions. Then there exists an entire rational function $f: V \to \mathbf{R}^n$ with $f|_{V_i} = f_i$.*

Proof: It suffices to prove this for $k = 2$. Write $f_i = p_i/q_i$ where p_i, q_i are polynomials with $q_i \neq 0$ on V_i, let $V_i = h_i^{-1}(0)$ for some polynomials h_i. Then

$$f = \frac{1}{h_1^2 + h_2^2}\left(\frac{p_2 q_2 h_1^2}{q_2^2 + h_2^2} + \frac{p_1 q_1 h_2^2}{q_1^2 + h_1^2}\right). \qquad \square$$

An important property of real algebraic sets is the complexification. For any real algebraic set $V \subset \mathbf{R}^n$ one can associate a complex algebraic set $V_{\mathbf{C}} \subset \mathbf{C}^n$ by taking the smallest complex algebraic set containing V (recall $\mathbf{R}^n \subset \mathbf{C}^n$). $\dim(V_{\mathbf{C}}) = 2\dim(V)$ as real algebraic sets. The complex conjugation on $V_{\mathbf{C}}$ induced from $\mathbf{C}^n$ defines an involution $j: V_{\mathbf{C}} \to V_{\mathbf{C}}$ with fixed point set V. This property imposes some topological restrictions on V. Any $x \in V$ has a well defined link $L(x) = S_\varepsilon \cap V$, where S_ε is a sphere of radius ε centered at x for a sufficiently small ε (recall algebraic sets are locally cone-like [M]). In [Su_1] Sullivan observed that for any $x \in V$ the Euler characteristic $\chi(L(x))$ of $L(x)$ is even. This follows from $\chi(L(x)) = \chi(L_{\mathbf{C}}(x)) = 0 \pmod 2$, where $L_{\mathbf{C}}(x)$ is the link of x in $V_{\mathbf{C}}$. The first equality holds since $L(x)$ is the fixed point set of the involution j on $L_{\mathbf{C}}(x)$, the second equality holds since $L_{\mathbf{C}}(x)$ is a stratified space with only odd dimensional strata. Algebraic sets are triangulated [Lo] and the local even Euler characteristic condition implies that the sum of k-simplexes of a compact k-dimensional algebraic set V^k is a cycle $[V] \in H_k(V; \mathbf{Z}/2\mathbf{Z})$ which we call the fundamental cycle. If V is connected then $H_k(V; \mathbf{Z}/2\mathbf{Z}) \cong \mathbf{Z}/2\mathbf{Z}$ and $[V]$ is the generator. This enables us to construct various polyhedra which can not be algebraic sets. For example let $X = S^1 \cup_f D^2$ where f is the degree 3 map $f: \partial D^2 \to S^1$. Then X can not even be homology equivalent to a 2-dimensional algebraic set since $H_2(X; \mathbf{Z}/2\mathbf{Z}) = 0$. The unreduced suspension Y of $\mathbf{RP}^2$ can not be homeomorphic to an algebraic set since it violates Sullivan's condition. However the reduced suspension $\bar{Y}$ of $\mathbf{RP}^2$, obtained from Y by collapsing an arc running from the north pole to the south pole, is homeomorphic to an algebraic set (since $\bar{Y}$ is an A_1-space, see §5). Hence unlike the first example Y is homotopy equivalent to an algebraic set.

Another useful property of real algebraic sets coming from complexification was observed by Benedetti and Tognoli [BT_1]. They noticed that if a closed smooth manifold M is a diffeomorphic image of a nonsingular algebraic set under an algebraic map, then $\bar{M} - M$ has dimension less than $\dim(M)$ where $\bar{M}$ denotes the Zariski closure of M. This can be easily generalized to:

LEMMA 0.2. *If $f: X \to \mathbf{R}^m$ is an entire rational function from an irreducible algebraic set such that $\chi(f^{-1}(x))$ is odd for a dense set of points $x \in f(X)$, then*

$$\dim(\overline{f(X)} - f(X)) < \dim f(X).$$

Proof: First replace X by the graph of f, then we can assume that $X \subset \mathbf{R}^n \times \mathbf{R}^m$ for some n and f is induced by the projection π to $\mathbf{R}^m$. By projectivizing we can replace $\mathbf{R}^n$ and $\mathbf{R}^m$ by $\mathbf{RP}^n$ and $\mathbf{RP}^m$ above (i.e. imbed them as charts).

Consider $X_{\mathbf{C}} \subset \mathbf{CP}^n \times \mathbf{CP}^m$ and let $\pi_{\mathbf{C}} : X_{\mathbf{C}} \to V$ be the map induced by the projection to $\mathbf{CP}^m$ and $V = \pi_{\mathbf{C}}(X_{\mathbf{C}})$. By algebraic Sard's theorem (3.7 of [Mu]) $\pi_{\mathbf{C}}$ is a fibre bundle map over the complement of a complex algebraic subset W of V. The real part of W has real codimension $\geqq 1$ in $\overline{\pi(X)}$. Therefore if $\dim(\overline{\pi(X)} - \pi(X)) \geqq \dim \pi(X)$ then we can find a point $x_0 \in (\overline{\pi(X)} - \pi(X)) \cap (V - W)$. Also by hypothesis we can find a point $x_1 \in \pi(X) \cap (V - W)$ with $\chi(\pi^{-1}(x_1))$ odd. The sets $\pi_{\mathbf{C}}^{-1}(x_0)$ and $\pi_{\mathbf{C}}^{-1}(x_1)$ are invariant by complex conjugation, and the fixed point sets of the involutions induced by the complex conjugation are the empty set and $\pi^{-1}(x_1)$, respectively. Hence $\chi(\pi_{\mathbf{C}}^{-1}(x_0)) = 0 \pmod 2$ and

$$\chi(\pi_{\mathbf{C}}^{-1}(x_1)) = \chi(\pi^{-1}(x_1)) = 1 \pmod 2;$$

this is a contradiction since $\pi_{\mathbf{C}}^{-1}(x_0) \approx \pi_{\mathbf{C}}^{-1}(x_1)$ (because π is a fibre bundle map over $V - W$). □

§1. Resolution of Algebraic Sets

Another important property of algebraic sets is the resolution property. This property forces algebraic sets to satisfy many topological conditions (see §5). Given an algebraic set V and an algebraic subset L; the *algebraic blowup of* V *along* L $B(V, L)$ defined to be the Zariski closure of

$$\{(x, \theta\, f(x)) \in V \times \mathbf{RP}^{n-1} \mid x \in V - L\},$$

where $f : (V, L) \to (\mathbf{R}^n, 0)$ is a polynomial whose coordinates generate $I(L)/I(V)$ and $\theta : \mathbf{R}^n - \{0\} \to \mathbf{RP}^{n-1}$ is the quotient map $\theta(x_1, \ldots, x_n) = [x_1 : \ldots : x_n]$. The amusing fact is that $B(V, L)$ is well defined algebraic subset of $V \times \mathbf{RP}^{n-1}$. Furthermore if V and L are nonsingular then $B(V, L)$ is diffeomorphic to the topological blowup of V along L $B_t(V, L) = (V - \text{interior } N) \cup E(N)$ where N is the normal disc bundle of L in V and $E(N)$ is the I-bundle over the projectivized normal bundle of L in V, i.e. $E(N)$ is obtained by replacing each fiber D^k of N by $\mathbf{RP}^k - \text{int}(D^k)$. There are natural projections π, π_t making the following commute

$$\begin{array}{ccc} B(V, L) & \xrightarrow{\ \pi\ } & \\ \wr\wr & & V \\ B_t(V, L) & \xrightarrow{\ \pi_t\ } & \end{array}$$

Given any polyhedron M with $L \subset M \subset V$ where L, V smooth manifolds then we define $B_t(M, L)$ to be the closure of $\pi_t^{-1}(M) - \pi_t^{-1}(L)$ in $B_t(V, L)$.