§4. Isolated Singularities

PROPOSITION 3.3. *Given* L, W, V *as above, then* $D_q(V) \cup L$ *is an algebraic subset of* $W \times \mathbf{R}^n$.

Proof: Let $p : \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}$ be an overt polynomial of degree e with $V = p^{-1}(0)$ and let q be as above. Define a polynomial $r : \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}$ by

$$r(x, y) = q(x)^e p\left(x, \frac{y}{q(x)}\right)$$

We claim $r^{-1}(0) = D_q(V) \cup L$. It is easy to see that

$$r^{-1}(0) \cap (W-L) \times \mathbf{R}^n = D_q(V) \cap (W-L) \times \mathbf{R}^n,$$

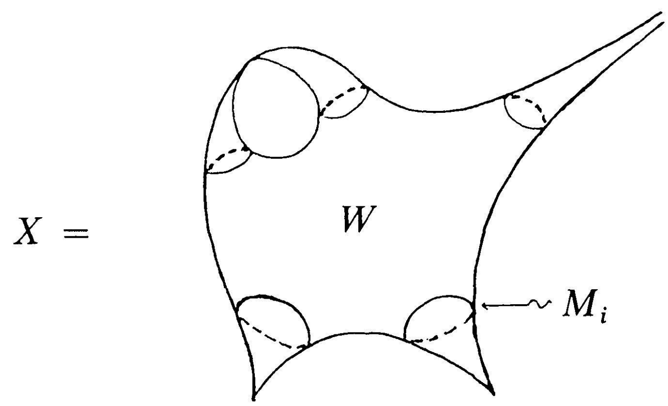
so it suffices to show that $r^{-1}(0) \cap (L \times \mathbf{R}^n) = L \times 0$. We decompose $p(x, y) = p_e(x, y) + \alpha(x, y)$ where $p_e(x, y)$ is homogeneous of degree e and $\alpha(x, y)$ is a polynomial of degree less than e. Hence if $(x, y) \in r^{-1}(0) \cap (L \times \mathbf{R}^n)$ then $r(x, y) = 0$ and $q(x) = 0$, which implies $r(x, y) = p_e(0, y) = 0$. Then $y = 0$ since p is overt, so $(x, y) \in L \times 0$. Conversely if $(x, y) \in L \times 0$ then $y = 0$ and $q(x) = 0$. Hence $r(x, y) = p_e(0, 0) = 0$, i.e. $(x, y) \in r^{-1}(0) \cap (L \times \mathbf{R}^n)$. □

There is a more useful version of Proposition 3.3 which says that after modifying D_q we can get $D_q(V) \cup L$ to be a projectively closed algebraic set (Proposition 3.1 of [AK_6]). This allows us to iterate this blowing down process.

§4. ISOLATED SINGULARITIES

The topology of real algebraic sets with isolated singularities is completely understood by the following Theorem.

THEOREM 4.1 ([AK_2]). *X is homeomorphic to an algebraic set with isolated singularities if and only if X is obtained by taking a smooth compact manifold W with boundary $\partial W = \bigcup_{i=1}^{r} M_i$, where each M_i bounds, then crushing some M_i's to points and deleting the remaining M_i's.*



One direction the proof follows from the resolution of singularities [H]. To prove it to the other direction we need the following:

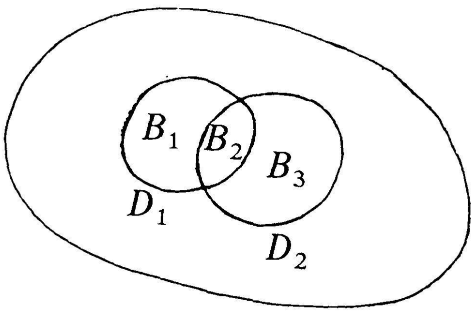
PROPOSITION 4.2. *If a closed smooth manifold* M *bounds a compact manifold, then it bounds a compact manifold* W *such that there are transversally intersecting closed smooth codimension one submanifolds* $W_1, \ldots, W_r$ *with* $W/\cup W_i \approx \mathrm{con}(M)$, *in other words* $\cup W_i$ *is a spine of* W.

Proof: Let $M = \partial Z$ where Z is some closed smooth manifold. Then pick balls D_i, $i = 1, 2, \ldots, r$ lying in interior (Z) such that:

(a) $\underset{i}{\cup} D_i$ is a spine of Z

(b) The spheres $S_i = \partial D_i$ intersect transversally with each other in Z

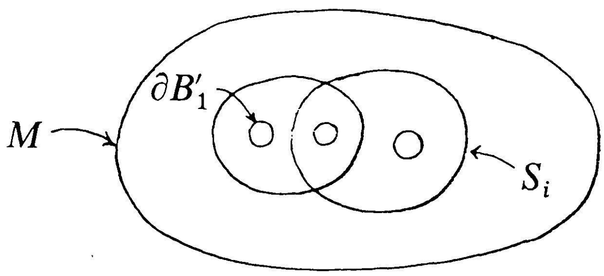
(c) $\cup D_i - \cup \partial D_i$ is a union of open balls $\bigcup_{j=1}^{s} B_j$.



Z

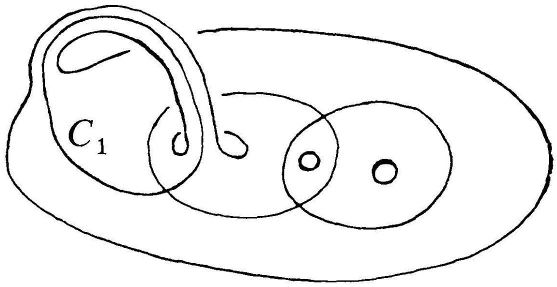
Let $B'_j \subset B_j$ denote a smaller ball. Then $Z_0 = Z - \bigcup_{j=1}^{s} \mathrm{interior}(B'_j)$ is a manifold with spine $\bigcup S_i$, and

$$\partial Z_0 = M \cup \bigcup_{j=1}^{s} \partial B'_j, \qquad \partial B'_j \approx S^m$$



Z_0

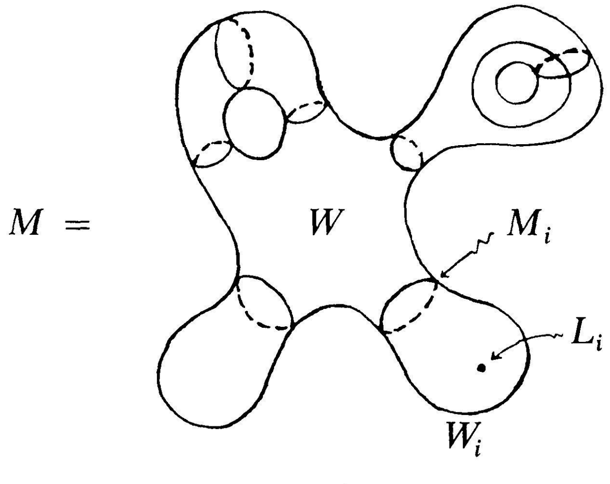
Order $\{B'_j\}$ so that there is an arc from M to $\partial B'_1$ intersecting exactly one S_i. Then attach a 1-handle to ∂Z_0 connecting M to $\partial B'_1$ get $Z_1 = Z_0 \cup$ (1-handle) as in the figure:



Z_1

Then $\partial Z_1 = M \cup \bigcup_{j=2}^{s} \partial B'_j$ and $\bigcup S_i \cup C_1$ is a spine of Z_1, where C_1 is the circle defined by the core of the 1-handle union of the arc. By continuing this fashion we get Z_s with $\partial Z_s = M$; and the spine of Z_s is transversally intersecting codimension one spheres and circles $\bigcup S_i \cup \bigcup_{j=1}^{s} C_j$. We are finished except C_j are not codimension one. We remedy this by topologically blowing up Z_s along $\bigcup C_j$, i.e. let $W = B(Z_s, \bigcup C_j)$ and let W_i to be the projectified normal bundles $P(C_j, Z_s)$ of C_j (i.e. the blown up circles), and $B(S_i, S_i \cap \bigcup C_j)$ we are done. □

Proof of Theorem 4.1: By Proposition 3.1 it suffices to prove this for one point compactification of X. Hence we can assume that X is compact. Let W be a compact smooth manifold, $\partial W = \bigcup_{i=1}^{r} M_i$ and each M_i bounds. By Proposition 4.2 we can assume $M_i = \partial W_i$ such that each W_i has a spine consisting of union of transversally intersecting codimension one closed smooth submanifolds L_i. Let $M = W \cup_{\partial} \bigcup W_i$



By Theorem 2.12 we can assume that the manifolds $(M; L_1, \ldots, L_r)$ are pairwise diffeomorphic to nonsingular algebraic sets $(Z; Z_1, \ldots, Z_r)$. Let $h : Z \to \mathbf{R}$ be an entire rational function with $h|_{Z_i} = i$ (h exists by Lemma 0.1). Let $\lambda : Z \to \mathbf{R}$ be a polynomial with $\lambda^{-1}(0) = \bigcup_i Z_i$. By Proposition 3.2 there exists a nonsingular projectively closed algebraic set $V \subset \mathbf{R}^2 \times \mathbf{R}^n$ and a birational diffeomorphism g making the following commute

$$\begin{array}{ccc} V & \hookrightarrow & \mathbf{R}^2 \times \mathbf{R}^n \\ {\scriptstyle g}\uparrow \approx & & \downarrow {\scriptstyle \pi} \\ Z & \underset{f}{\rightarrow} & \mathbf{R}^2 \end{array}$$

where $f = (h, \lambda)$. Let $L = \{(1, 0), (2, 0), \ldots, (r, 0)\}$ then by Proposition 3.3 we can blow down V over L algebraically. This gives an algebraic set homeomorphic to X. □

COROLLARY 4.3. *Up to diffeomorphism nonsingular algebraic sets are exactly the interiors of compact smooth manifolds with boundary (possibly empty).*

The following is a local knottedness theorem of real algebraic sets. It is an ambient version of Theorem 4.1. It says that unlike complex algebraic sets all knots can occur as links of singularities.

THEOREM 4.4 ([AK$_4$]). *Let W^m be a compact smooth submanifold of S^{n-1} imbedded with trivial normal bundle with codimension $\geqq 1$. Then there exists an algebraic set $V \subset \mathbf{R}^n$ with $\mathrm{Sing}(V) = \{0\}$ such that $(B_\varepsilon, B_\varepsilon \cap V) \approx (B^n, \mathrm{cone}(\partial W))$ for all small $\varepsilon > 0$, where B_ε is the ball of radius ε centered at 0. In fact $\varepsilon(\partial W)$ is isotopic to $\partial B_\varepsilon \cap V$ in ∂B_ε.*

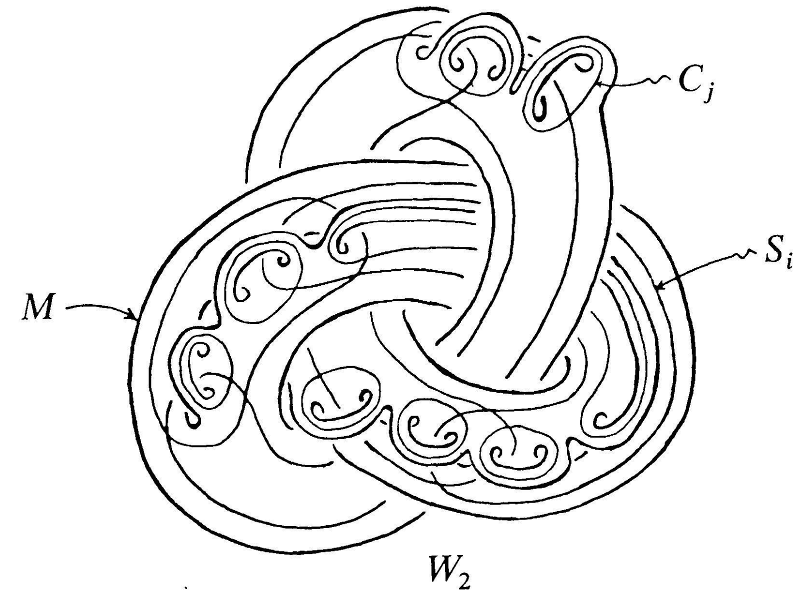
By taking W to be the Seifert surface of a knot we get an interesting fact.

COROLLARY 4.5. *Any knot $K^{n-3} \subset S^{n-1}$ is isotopic to a link of an algebraic set V in $\mathbf{R}^n$.*

A sketch proof of Theorem 4.4: First identify $W \subset \mathbf{R}^{n-1} \approx S^{n-1} - \infty$, and call $M = \partial W$. Then apply the process of getting nice spines to W^m (Proposition 4.2); i.e. pick a family of discs D_i, $i = 1, \ldots, r$ in W whose boundaries are in general position, and $W/\cup D_i \approx \mathrm{cone}(M)$ and $\bigcup D_i - \bigcup S_i$ is a disjoint union of open balls $\bigcup B_j$ where $S_i = \partial D_i$. Let W_1 be the manifold obtained by removing a small open ball from each B_j. Now by attaching 1-handles to W_1 as in

Proposition 4.2 we obtain W_2, whose spine consists of $\bigcup S_i$ union circles $\bigcup C_j$, with $\partial W_2 = M$.

Observe that this whole process can be done inside $\mathbf{R}^{n-1}$ and C_j and S_i are unknotted in $\mathbf{R}^{n-1}$



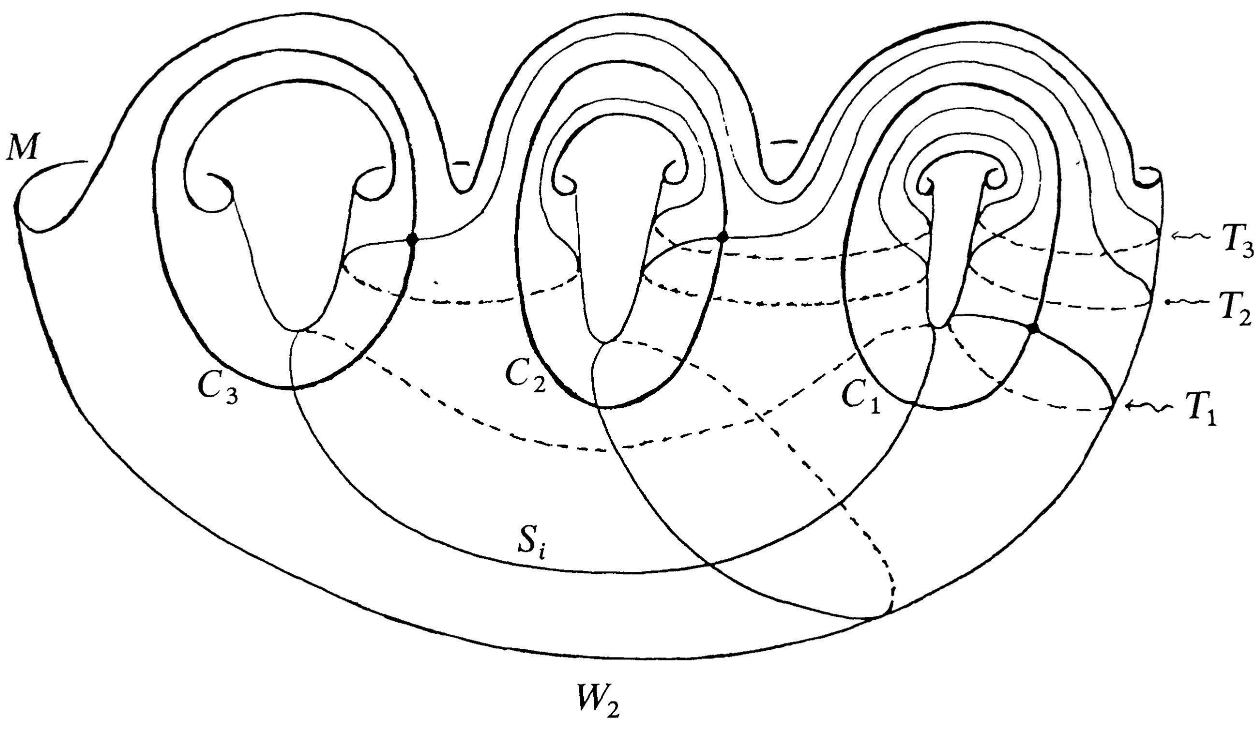
We claim that there is disjointly imbedded $m - 1$ spheres $T_j, j = 1, \ldots, s$ in W_2 such that

(1) Each T_j is unknotted in $\mathbf{R}^{n-1}$.

(2) Each T_j meets C_j at a single point, and $T_j \cap C_i = \emptyset$ for $i \neq j$.

(3) For each i there is $B_i \subset \{1, 2, \ldots, s\}$ so that $S_i \cup \bigcup_{j \in B_i} T_j$ separates W_2.

This can be easily done as in the following picture.



(1) and (2) are easily checked from the picture. To see (3), let $B_i = \{j \mid C_j \cap S_i \neq \emptyset\}$.

Let $W_3 = W_2 \underset{\partial}{\cup} - W_2$. The imbedding $W_2 \subset \mathbf{R}^{n-1}$ can be extended to an imbedding of W_3. Since T_j and C_j are unknotted and by (2), we can isotop W_3 so that $T_j \cup C_j$ in W_3 coincides with $S^{m-1} \cup S^1$ in $(S^{m-1} \times S^1)_j$, where $(S^{m-1} \times S^1)_j$, $j = 1, \ldots, s$ are disjointly imbedded copies of the standard $S^{m-1} \times S^1$ in $\mathbf{R}^{n-1}$. We can assume that some open neighborhoods of these sets in W_3 and $(S^{m-1} \times S^1)_j$ also coincide. By Theorem 2.3 and Remark 2.4 we can isotop W_3 to a component of a nonsingular algebraic set Z fixing $T_j \cup C_j$ for all j. In fact after a minor adjustment (to proof of Theorem 2.3) we can assume that Z is projectively closed. Continue to call isotoped copy of S_i by S_i.

Since as codimension one homology classes $[S_i] = [\bigcup_{j \in B_i} T_j]$ and $\bigcup T_j$ is a nonsingular algebraic set, S_i can be made a nonsingular algebraic set for each i (Theorem 2.6). Hence the spine $L = \bigcup S_i \cup \bigcup C_j$ of $W_2 \subset Z$ can be assumed to be an algebraic set. Since Z is projectively closed so is L.

Let p, q be overt polynomials with $p^{-1}(0) = Z$ and $q^{-1}(0) = L$. Define

$$V = \{(x, t) \in \mathbf{R}^{n-1} \times \mathbf{R} \mid t^{2e+1} = q^*(x, t)^2,\ p^*(x, t) = 0\}$$

where $p^*(x, t) = t^d p(x/t)$, $q^*(x, t) = t^e q(x/t)$ where $d =$ degree p, $e =$ degree q. If $(x, t) \in V$ then $t \geqq 0$; and if $t = 0$ then $x = 0$ since p is overt.

$$(\mathbf{R}^{n-1} \times \varepsilon, (\mathbf{R}^{n-1} \times \varepsilon) \cap V) \approx (\mathbf{R}^{n-1}, q^{-1}(\varepsilon) \cap Z) \approx (\mathbf{R}^{n-1}, M),$$

since $q^{-1}(\varepsilon) \cap Z \approx \partial W_2 = M$. We are almost done.

Let $S_\varepsilon^{n-1} = \{(x, t) \in \mathbf{R}^{n-1} \times \mathbf{R} \mid |x|^2 + t^2 = \varepsilon^2\}$, and $\varphi_\varepsilon \colon \mathbf{R}^{n-1} \to S_\varepsilon^{n-1}$ be the imbedding $\varphi_\varepsilon(y) = (1+|y|^2)^{-1/2}(\varepsilon y, \varepsilon)$. Then

$$\varphi_\varepsilon^{-1}(S_\varepsilon^{n-1} \cap V) = \{y \in \mathbf{R}^{n-1} \mid p(y) = 0,\ \ q^4(y)\,(1+|y|^2) = \varepsilon^2\}$$

which is isotopic to M in $\mathbf{R}^{n-1}$ for all small $\varepsilon > 0$. Hence $(S_\varepsilon^{n-1}, S_\varepsilon^{n-1} \cap V) \approx (S^{n-1}, M)$ for all small $\varepsilon > 0$. □