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MILNOR LATTICES AND GEOMETRIC BASES OF SOME SPECIAL SINGULARITIES [1])

by Wolfgang EBELING

INTRODUCTION

The subject of this note are certain invariants associated to the topology of a complex hypersurface singularity $f : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0), n \equiv 3(4)$, which are defined via the Milnor fibration and deformation theory. These invariants are the homology group of the Milnor fiber of middle dimension $n - 1$ as an abelian group (determined by the Milnor number μ), the signature (μ_0, μ_+, μ_-) of the intersection form, the homology group of dimension $n - 2$ of the link of the singularity as an abelian group, the linking form, the intersection form, weakly distinguished bases, distinguished bases. The order reflects the relative strength of these invariants, but it is not always a strict order: the knowledge of the intersection form turns out to be equivalent to the knowledge of the invariants listed before.

The aim of this note is to give a survey of some recent results on these invariants for special classes of singularities. In [7] we studied some of these invariants for the singularities of Arnold's lists. Here we summarize these results and give additional information. We also consider another class of singularities, namely the minimally elliptic hypersurface singularities as studied by Laufer [14]. We state some general features about the above invariants for these two classes of singularities. One of the problems is to find a normal form of the Dynkin diagrams with respect to bases of a certain type for a whole class of singularities. We indicate such a form with respect to weakly distinguished bases for the minimally elliptic hypersurface singularities. But it turns out that all the above invariants except the class of distinguished bases are to weak to distinguish between singularities of different topological type. We also discuss a canonical form of Dynkin diagrams with respect to distinguished bases for the uni- and bimodular singularities, which are contained in both classes of singularities mentioned above.

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[1]) This article has already been published in *Nœuds, tresses et singularités*, Monographie de l'Enseignement Mathématique N° 31, Genève 1983, p. 129-146.

The results were obtained by the following method:

a) Find a distinguished basis for the given singularity. This is done using methods of Gabrielov, especially [11]. But the Dynkin diagrams of these bases are very complicated and contain many cycles.

b) Transform this diagram into a "nicer" form, where the information one is looking for is more transparent.

c) Analyse this diagram.

The paper is organized as follows. In section 1 + 2 we recall the definitions of the invariants and the basic relations among them and discuss the admissible transformations for b) above. Section 3 is devoted to a study of the weaker invariants including weakly distinguished bases of the above mentioned singularities. In section 4 we consider distinguished bases of the uni- and bimodular singularities.

There are also other invariants associated to singularities such as e.g. the monodromy groups. For a discussion of the relations among the various invariants and of their relative strength with respect to geometrical problems we refer to the expository article of E. Brieskorn [3]. For a description of the monodromy groups we refer to [8, 9]. There are also other interesting phenomena related to the above invariants in the class of bimodular singularities such as an extension of Arnold's strange duality. This is the subject of a joint paper with C. T. C. Wall, which is in preparation.

This paper is an extended version of the talk given by the author at the conference on the "Topology of complex singularities" at Les Plans-sur-Bex/Switzerland, March 27-April 2, 1982. The author thanks the organizers of this meeting, especially C. Weber, for their invitation. He is also grateful to E. Brieskorn for helpful discussions, which influenced the presentation in section 2. The final work on this paper was carried out at the State University of Utrecht and was supported by the Netherlands Foundation for Mathematics S.M.C. with financial aid from the Netherlands Organization for the Advancement of Pure Research (Z.W.O.). The author thanks these institutions for their hospitality.

1. The Milnor Lattice of a Singularity

Let $f : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ be the germ of an analytic function with an isolated singularity at 0. Let B_ε denote an open ball of radius ε in $\mathbf{C}^n$ around 0. Then for sufficiently small $\delta > 0$ and $\varepsilon > 0$

$$V_\delta = f^{-1}(\delta) \cap B_\varepsilon$$

is the Milnor fiber of f. The Milnor fiber has the homotopy type of a bouquet of $n-1$-spheres, and therefore its only interesting homology group is

$$L = H_{n-1}(V_\delta, \mathbf{Z}),$$

a free $\mathbf{Z}$-module of rank μ. We shall assume throughout this paper that $n \equiv 3(4)$. Then the intersection form $\langle\, ,\,\rangle$ on L is symmetric and satisfies $\langle x, x\rangle \in 2\,\mathbf{Z}$ for all $x \in L$. Therefore L provided with $\langle\, ,\,\rangle$ is an even lattice, which we call the *Milnor lattice* of f. To L is associated a triple of numbers (μ_0, μ_+, μ_-), where μ_0, μ_+, μ_- is the number of 0's, 1's, -1's respectively on the diagonal after a diagonalisation of the quadratic form over the real numbers.

To L is also associated another invariant, which we define next. Let ker L denote the kernel of L and $\bar{L} = L/\ker L$ the corresponding nondegenerate lattice. Let $\bar{L}^\# = \mathrm{Hom}(\bar{L}, \mathbf{Z})$ be the dual lattice of $\bar{L}$, and $G_L = \bar{L}^\#/\bar{L}$. Then G_L is a finite abelian group of order $|\operatorname{disc} \bar{L}|$. The bilinear form on L induces a bilinear form

$$b_L \colon G_L \times G_L \to \mathbf{Q}/\mathbf{Z}$$

on G_L defined by $b_L(\bar{u}, \bar{v}) = \langle u, v\rangle \pmod 1$ for $u, v \in \bar{L}^\#$. This form is called *discriminant bilinear form.* Since L is even, there is also an induced quadratic form

$$q_L \colon G_L \to \mathbf{Q}/2\mathbf{Z}$$

defined by $q_L(\bar{u}) = \langle u, u\rangle \pmod 2$ for $u \in \bar{L}^\#$. It is called *discriminant quadratic form.* The pair (G_L, b_L) can be interpreted geometrically as follows: Let K denote the link of the singularity f, which is equal to the boundary of $\bar{V}_\delta$,

$$K = \partial \bar{B}_\varepsilon \cap f^{-1}(0) = \partial \bar{V}_\delta\,.$$

Then G_L is the torsion subgroup of $H_{n-2}(K, \mathbf{Z})$ and b_L is the classical linking form (cf. [4]). Moreover μ_0 is equal to the rank of $H_{n-2}(K, \mathbf{Z})$.

There are results in the theory of quadratic forms of Durfee, Kneser, Nikulin and Wall, that the above invariants are already sufficient to determine the isomorphism class of the lattice, if the lattice satisfies certain conditions, in particular is indefinite. From these results V. V. Nikulin has derived the following theorem (cf. [17]).

THEOREM 1.1. *The Milnor lattice L is determined as an abstract lattice by (μ_0, μ_+, μ_-) and the discriminant bilinear form b_L (resp. the discriminant quadratic form q_L).*

2. Geometric Bases of the Milnor Lattice

There are certain classes of bases for the Milnor lattice, which are distinguished by the geometry of the singularity. We shall recall the definition of these bases (see [12] for more information). For that purpose we shall consider the semiuniversal deformation of the singularity.

Let $X_0 = f^{-1}(0)$ be the complex analytic hypersurface defined by f. Let $F: X \to S$ be a suitable representative of the semi-universal deformation of the germ $(X_0, 0)$. Denote by $D \subset S$ the corresponding discriminant, i.e. the image of the critical locus of F. Put $S' = S - D$, $X' = F^{-1}(S')$ and $F' = F|_{X'}$. Then $F': X' \to S'$ is a C^∞-fiber bundle, where each fiber is diffeomorphic to the Milnor fiber.

Choose a generic complex line in the affine space containing S, which intersects S in a complex disc Δ. Then Δ intersects the discriminant D in μ points $c_1, \ldots, c_\mu$, which lie in the interior of Δ. Choose a basepoint s_0 on the boundary of Δ. Let X_{s_0} denote the fiber of F over s_0. Then $H_{n-1}(X_{s_0}, \mathbf{Z})$ is isomorphic to L and will also be denoted by L. We shall construct bases of L. Each path ϕ_i in $\Delta' = \Delta - \{c_1, \ldots, c_\mu\}$ from c_i to s_0 determines an element of L as follows. The fiber over c_i has only one singular point which is an ordinary double point. Near this point, the fibers are given locally by an equation

$$z_1^2 + \ldots + z_n^2 = r^2 .$$

The real sphere

$$S^{n-1} = \{(x_1, \ldots, x_n) \mid x_1^2 + \ldots + x_n^2 = r^2\}, \quad z_i = x_i + iy_i ,$$

represents after the choice of an orientation a homology class in a fiber over a point of ϕ_i near c_i. Transport along the path ϕ_i gives an element $e \in L$ satisfying $\langle e, e \rangle = -2$. Such an element is called *vanishing cycle*. Let Λ^* denote the set of vanishing cycles. Choosing a path ϕ_i for each c_i yields a system of μ vanishing cycles.

In order to get a basis of vanishing cycles, there are several possible restrictions on the choice of paths. In order to define these restrictions and the corresponding classes of bases, we need the notion of a *simple loop* corresponding to a path ϕ_i from c_i to s_0. This is the element of $\pi_1(\Delta', s_0)$ represented by the loop τ_i going from s_0 to a point sufficiently near to c_i along the path ϕ_i, going once around c_i in the positive direction (counterclockwise) and returning to s_0 along the path ϕ_i. This loop induces an automorphism of L, the Picard-Lefschetz-transformation corresponding to the path ϕ_i. It is given by the reflection s_{e_i} on the orthogonal complement of the vanishing cycle e_i corresponding to ϕ_i, i.e. s_{e_i} is defined by

$$s_{e_i}(x) = x + \langle x, e_i \rangle e_i \quad \text{for } x \in L .$$

1) *Distinguished Bases*

Choose the paths ϕ_i non-selfintersecting, let any two have only s_0 as a common point and number the paths in the order in which they arrive at s_0, counted clockwise beginning from the boundary of Δ. Then the corresponding system of vanishing cycles forms a basis, and a basis of L obtained in this way is called a *distinguished basis*. Let $\mathscr{B}^*$ denote the set of all distinguished bases of L.

On $\mathscr{B}^*$ there is an operation of the braid group Z_μ in μ strings, where a generator α_i operates as follows: Let $B = \{e_1, ..., e_\mu\} \in \mathscr{B}^*$ be a distinguished basis defined by the system of paths $\{\phi_1, ..., \phi_\mu\}$. The operation α_i is induced by the following elementary operation on the level of paths (cf. Fig. 1):

$$(\phi_1, ..., \phi_\mu) \to (\phi_1, ..., \phi_{i-1}, \phi_{i+1}\tau_i, \phi_i, \phi_{i+2}, ..., \phi_\mu) .$$

The operation α_i is then given by

$$(e_1, ..., e_\mu) \to (e_1, ..., e_{i-1}, s_{e_i}(e_{i+1}), e_i, e_{i+2}, ..., e_\mu) .$$

The inverse operation α_i^{-1} is also denoted by β_{i+1}.

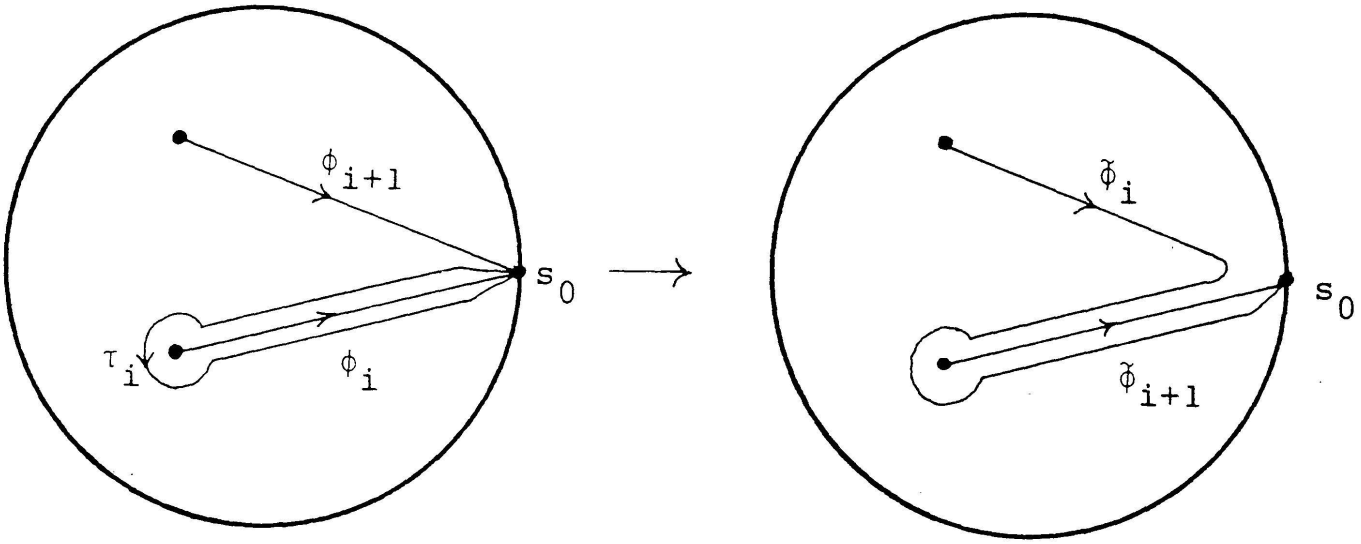


FIGURE 1

A distinguished basis does not only depend on the choice of paths, but also on the choice of orientation of the cycles. The change of orientation of a cycle e_i corresponds to an operation γ_i given by $\gamma_i(e_j) = e_j$ for $j \neq i$ and $\gamma_i(e_i) = -e_i$.

Therefore there is in addition an operation of $(\mathbf{Z}/2\mathbf{Z})^\mu$ on $\mathcal{B}^*$. Together one has an operation of the semidirect product

$$Z^* = Z_\mu \rtimes (\mathbf{Z}/2\mathbf{Z})^\mu$$

on $\mathcal{B}^*$ (both factors considered as subgroups of the symmetric group of $\mathcal{B}^*$).

PROPOSITION 2.1. *The operation of Z^* on $\mathcal{B}^*$ is transitive.*

For a proof see [12].

PROPOSITION 2.2. *If $\mathcal{B}^*$ contains a basis $B = \{e_1, \ldots, e_\mu\}$ with $\langle e_i, e_j \rangle \in \{0, 1, -1\}$ for $i \neq j$, then already Z_μ operates transitively on $\mathcal{B}^*$.*

Proof. It suffices to show that any basis $\tilde{B} \in \mathcal{B}^*$ can be transformed into B by Z_μ. By the previous proposition there exists an element of Z^* which transforms $\tilde{B}$ to B. We show that for each i there exists an element $\sigma_i \in Z_\mu$ such that $\gamma_i(B) = \sigma_i(B)$. If $\langle e_j, e_{j+1} \rangle = \varepsilon$, $\varepsilon = \pm 1$, then $\alpha_j^{12} = id$ and

$$\alpha_j^{3\varepsilon}(B) = \gamma_{j+1}(B), \ \alpha_j^{-3\varepsilon}(B) = \gamma_j(B) .$$

Now let k be the smallest integer such that $\langle e_i, e_{i+k} \rangle \neq 0$, $k \neq 0$. We consider the case $k > 0$, the case $k < 0$ is analogous. If $k = 1$, we apply the previous remark. Otherwise the transformation $(\alpha_{i+k-2} \circ \ldots \circ \alpha_i)(B)$ interchanges e_i and e_{i+k-1} and leaves all other basis elements fixed. Hence one can now apply the above remark. This proves the proposition.

Let $B = \{e_1, \ldots, e_\mu\}$ be a distinguished basis. Then

$$h = s_{e_1} \circ s_{e_2} \circ \ldots \circ s_{e_\mu}$$

is the *classical monodromy operator* of the singularity.

2) *Weakly Distinguished Bases*

We now impose the only condition on the system of paths that the corresponding simple loops generate $\pi_1(\Delta', s_0)$. Then it can be shown that the corresponding system of vanishing cycles forms again a basis, and a basis obtained in this way is called a *weakly distinguished basis*. Let $\mathcal{B}^0$ denote the set of all weakly distinguished bases.

Since the numbering does not play a role for a weakly distinguished basis, we have on $\mathcal{B}^0$ also an operation of the symmetric group $\mathcal{S}_\mu$ of degree μ. Let Z^0 be the group generated by Z^* and $\mathcal{S}_\mu$. There are some special elements in Z^0 defined as follows: Let $B = \{e_1, ..., e_\mu\}$ be a weakly distinguished basis defined by a system of paths $\{\phi_1, ..., \phi_\mu\}$. Let $\{\tau_i\}$ be the corresponding system of simple loops. For $i \neq j$ we define a transformation $\alpha_i(j)$ on the different levels as follows:

$$(\phi_1, ..., \phi_\mu) \to (\phi_1, ..., \phi_{j-1}, \phi_j\tau_i, \phi_{j+1}, ..., \phi_\mu),$$

$$(\tau_1, ..., \tau_\mu) \to (\tau_1, ..., \tau_{j-1}, \tau_i^{-1}\tau_j\tau_i, \tau_{j+1}, ..., \tau_\mu),$$

$$(e_1, ..., e_\mu) \to (e_1, ..., e_{j-1}, s_{e_i}(e_j), e_{j+1}, ..., e_\mu).$$

The inverse transformation is denoted by $\beta_i(j)$. In the case $n \equiv 3(4)$ it coincides with $\alpha_i(j)$ on the homology level. Now Z^0 is generated by the transformations $\alpha_i(j)$, $\mathcal{S}_\mu$ and change of orientation operations.

CONJECTURE 2.3 (Gusein-Zade). *The operation of Z^0 on $\mathcal{B}^0$ is transitive.*

This conjecture can be reduced to a problem in pure combinatorial group theory, see [12]. It is not known to the author, whether the conjecture is true.

The *monodromy group* Γ of the singularity f is the image of $\pi_1(\Delta', s_0)$ under the natural representation

$$\rho: \pi_1(\Delta', s_0) \to \mathrm{Aut}(L).$$

It is generated by the reflections corresponding to the elements of a weakly distinguished basis.

The matrix of the bilinear form on L with respect to a basis B of vanishing cycles is described in the usual way by a graph with (possibly multiple) edges weighted by $+1$ or -1, where we indicate negative weight by a dotted line. This graph is called the *Dynkin diagram* with respect to B.

3. MILNOR LATTICES AND WEAKLY DISTINGUISHED BASES OF SOME SPECIAL SINGULARITIES

We shall consider the Milnor lattices of some specific singularities, namely the singularities of Arnold's lists and the minimally elliptic hypersurface singularities. By the singularities of Arnold's lists we mean the singularities, for which Arnold has given normal forms in [1], i.e. the singularities of the series A, D, J, E, X, Y, Z, W, T, Q, S, U and V. Most of these series contain singularities with arbitrary number of moduli. The minimally elliptic hypersurface singularities can be defined as follows (cf. [5]): They are the singularities f:

$(\mathbf{C}^3, 0) \to (\mathbf{C}, 0)$ with $\mu_0 + \mu_+ = 2$. They have been classified by Laufer (cf. [14]). Both classes of singularities contain in particular all uni- and bimodular singularities.

By the methods of [11] one can show that all the above singularities have a distinguished basis $B = \{e_1, ..., e_\mu\}$ satisfying $\langle e_i, e_j \rangle \in \{0, 1, -1\}$ for $i \neq j$, i.e. satisfy the conditions of Prop. 2.2. Using the operation of Z^0, we look for other elements of the sets $\mathscr{B}^0$, which reveal more of the structure of the Milnor lattice. In [7] we have listed weakly distinguished bases for the singularities of Arnold's lists except the series V, which give rise to certain orthogonal splittings of the corresponding Milnor lattices. From these results we also derived that the monodromy groups of almost all of these singularities can be characterized arithmetically, which is even true for a much larger class of singularities (cf. [8, 9]). The orthogonal splittings enable one to compute in an easy way the discriminant quadratic forms of the corresponding Milnor lattices. In particular one gets the following result. Let $\lambda(G_L)$ denote the minimal number of generators of G_L.

THEOREM 3.1. *The following is true for all singularities of Arnold's lists:*

(i) $\mu_0 \leqslant 2$, $\mu_- \geqslant 5(\mu_0 + \mu_+) - 4$.

(ii) *The number* $\mu_0 + \mu_+$ *grows proportional to the number of moduli within each series.*

(iii) $\lambda(G_L) \leqslant 3$.

For the minimally elliptic hypersurface singularities one can derive the following result. We first define a graph-theoretical invariant. Let H be a graph. For a vertex $v \in H$, the degree of v, $\deg v$, is the number of edges incident with v. Let $z(H)$ be the number of cycles of H of the form $v_0, v_1, ..., v_r = v_0$, where there exists a number k, $1 \leqslant k < r$, with $\deg v_i \geqslant 3$ for $1 \leqslant i \leqslant k$ and $\deg v_i = 2$ otherwise. Define

$$\sigma(H) = \sum_{\substack{v \in H \\ \deg v \geqslant 3}} (\deg v - 2) + z(H).$$

THEOREM 3.2.

(i) *Let* f *be a minimally elliptic hypersurface singularity with* $\mu_+ = 2$ (*hence* $\mu_0 = 0$). *Then there exists a weakly distinguished basis* $B = \{e_1, ..., e_\mu\}$ *of* f *satisfying the following properties:*

a) $\langle e_1, e_2 \rangle = 1$,

$\langle e_1, e_i \rangle = 0$, $\langle e_2, e_i \rangle = \langle e_3, e_i \rangle$ *for* $3 \leqslant i \leqslant \mu$.

b) *For* $i, j \in \{3, ..., \mu\}, i \neq j, \langle e_i, e_j\rangle \in \{0, 1\}$,
(The matrix $(-\langle e_i, e_j\rangle)_{3 \leqslant i, j \leqslant \mu}$ *is therefore an indecomposable symmetric Cartanmatrix of negative type in the sense of* [13]).

c) *Let* H *denote the subgraph of the Dynkin diagram with respect to* $\{e_3, ..., e_\mu\}$. *Then*

$$1 \leqslant \sigma(H) \leqslant 4\,.$$

(ii) *For all minimally elliptic hypersurface singularities*

$$\lambda(G_L) \leqslant 4\,.$$

More information about the Milnor lattices of these singularities will be given in a forthcoming paper. Dynkin diagrams corresponding to weakly distinguished bases satisfying (i) are given for the unimodular singularities in [10] (here $\sigma(H)=1$) and for the bimodular in [7].

THEOREM 3.3. *For each bimodular singularity* $\mu_0 = 0, \mu_+ = 2, \lambda(G_L) \leqslant 3$. *Moreover there exists a weakly distinguished basis satisfying Th. 3.2* (i) *with* $\sigma(H) = 2$.

Example. Consider the following two bimodular families of singularities:

$$E_{18}: x^3 + y^{10} + z^2 + a_0 xy^7 + a_1 xy^8$$
$$Q_{18}: x^3 + yz^2 + y^8 + a_0 xy^6 + a_1 xy^7\,.$$

No member of the class E_{18} is topologically equivalent to a member of the class Q_{18}, since the resolution graphs are different and by Neumann's result [16] the corresponding links are not diffeomorphic. This implies in particular that the corresponding Milnor fibers are not diffeomorphic. But the singularities of both families have the same discriminant 3 and the same signature $(\mu_0, \mu_+, \mu_-) = (0, 2, 16)$. By a result in the theory of quadratic forms [7, Satz 2.2], there is up to isomorphism only one lattice with these invariants. So both singularity classes have the same Milnor lattice L, and an explicit description of L is e.g. given by

$$L = E_6 \perp E_8 \perp U \perp U\,.$$

Moreover the sets $\mathscr{B}^0$ coincide in both cases, implying also that the monodromy groups Γ are the same. Dynkin diagrams with respect to weakly distinguished bases satisfying the conditions of Theorem 3.2 are given by the graph of Fig. 2, where the quintuples (a, b, c, d, e) are listed in Table 1. These are also the only possibilities of a graph of the form of Fig. 2 to be a graph of the above lattice L. The graph of Fig. 2 satisfies $\sigma = 2$.

However, the sets $\mathscr{B}^*$ are different for the two classes of singularities, because the classical monodromy operators have different orders, namely 30 for E_{18} and 48 for Q_{18}.

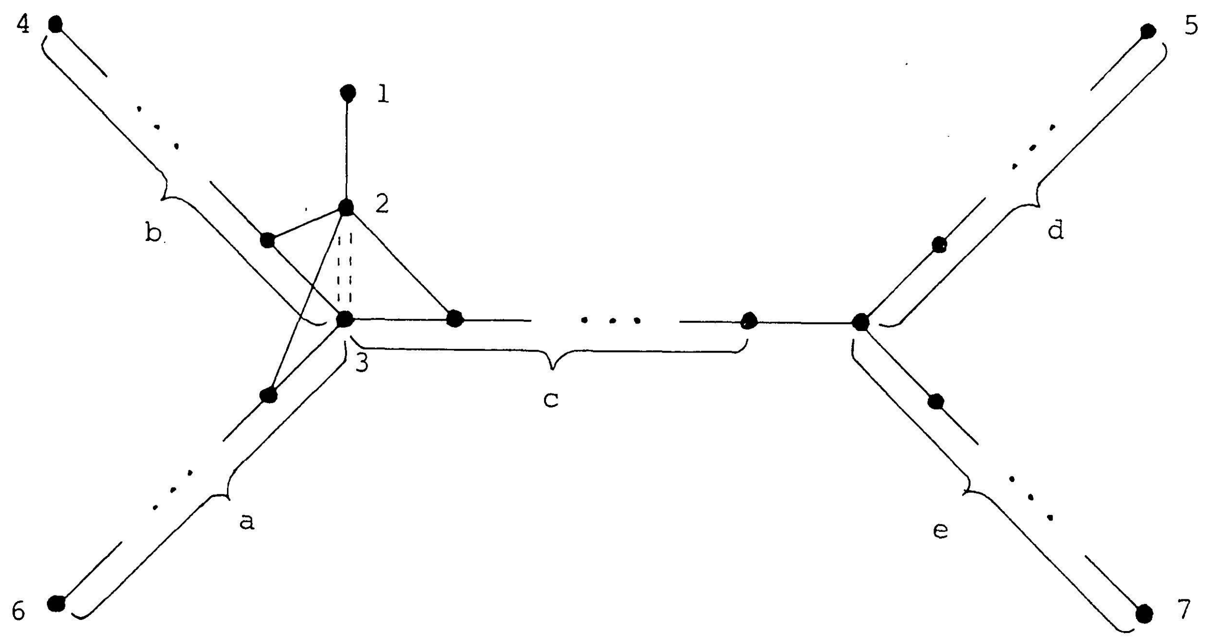


FIGURE 2

TABLE 1

a	b	c	d	e
2	3	9	2	3
2	3	8	3	3
2	5	6	3	3
3	3	5	3	5

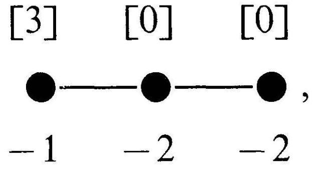
Remark. There are also examples of singularities with different numbers of moduli which have isomorphic L, Γ and $\mathscr{B}^0$. Moreover J. Wahl has informed me that H. Laufer has found an example of two singularities of different topological type which even have the same resolution graph and hence diffeomorphic links, isomorphic Milnor lattices and by [8] isomorphic monodromy groups. These are the singularities given by

$$z^3 + x^4 + y^{36}$$

and

$$z^2 + y(x^{12} + y^{18}).$$

The resolution graph is in both cases



where the number in brackets denotes the genus, the other the selfintersection number of the corresponding cycle. Here $(\mu_0, \mu_+, \mu_-) = (6, 42, 162)$. However, the orders of the classical monodromy operators are 36 resp. 38.

4. Distinguished Bases for the Bimodular Singularities

We have seen in the last section that there are bimodular singularities which have the same Dynkin diagrams with respect to weakly distinguished bases, but not with respect to distinguished bases. We now turn our attention to the sets $\mathscr{B}^*$ for these singularities. Let us first look at the unimodular case. All exceptional unimodular singularities have a weakly distinguished basis with a Dynkin

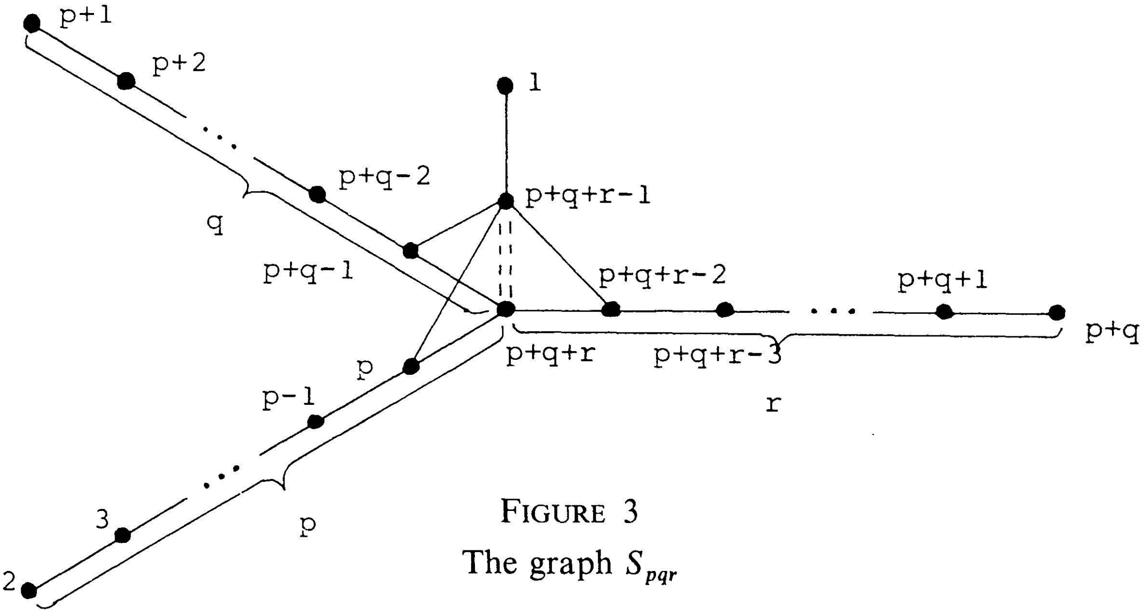


Figure 3
The graph S_{pqr}

diagram in Gabrielov's canonical form given by the graph of Fig. 2 setting $d = e = 1$. One can show that these graphs provided with the numbering of Fig. 3 also correspond to distinguished bases. (The graph with this numbering is obtained from the graph in [7, Abb. 15] by the following transformations: We indicate only the transformations for the first branch, the other branches are treated in an analogous manner: $\beta_7, \beta_6, \beta_5, \beta_4, \beta_3$; $\beta_8, \beta_7, \beta_6, \beta_5, \beta_4$; ...; $\beta_{p+4}, \beta_{p+3}, \beta_{p+2}, \beta_{p+1}, \beta_p$; $\gamma_2, \gamma_3, \ldots, \gamma_{p-1}$). We call this graph S_{pqr}.

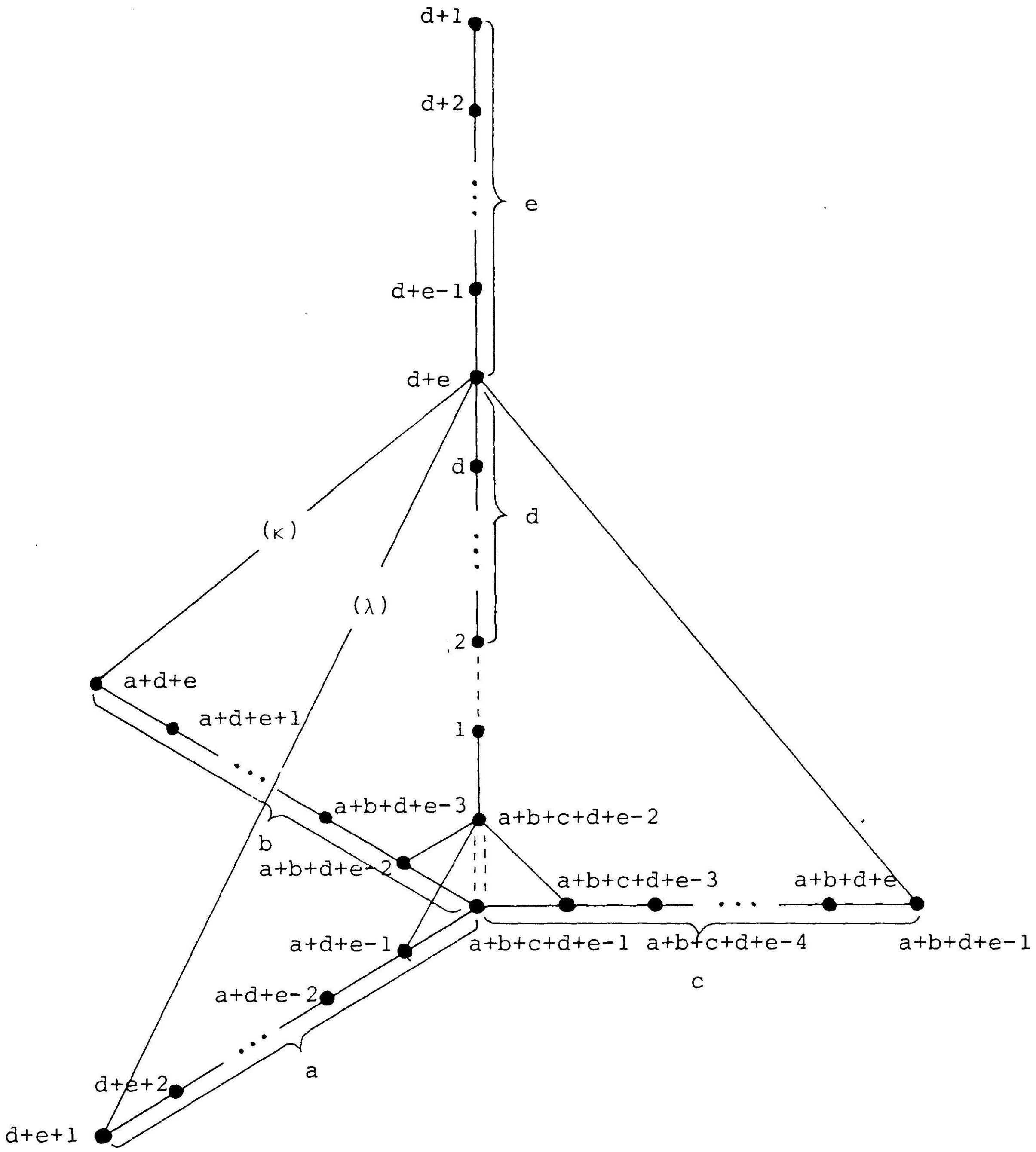


FIGURE 4
The graph $R^{\kappa\lambda}_{abcde}$

A natural form for the Dynkin diagrams of elements of $\mathscr{B}^0$ for the bimodular singularities E_{18} and Q_{18} is given in Fig. 2. Not all bimodular singularities have a Dynkin diagram of this type, one has to allow additional edges between e_4 and e_5 and between e_6 and e_7 (see [7]). But one can show by the methods introduced later in this section that none of the diagrams of Fig. 2/Table 1 equipped with any numbering corresponds to a distinguished basis of any of these singularities. However, there are elements of $\mathscr{B}^*$ with a Dynkin diagram of a form which is very close to the form of Fig. 2: one has to add only one dotted edge to this diagram. More precisely we have the following theorem:

THEOREM 4.1. *All bimodular singularities have a distinguished basis with the Dynkin diagram* $R^{\kappa\lambda}_{abcde}$ *shown in Fig. 4, where the values* $\kappa, \lambda, a, b, c, d, e$ *are given in Table 2.*

The graph $R^{\kappa\lambda}_{abcde}$ is defined for $a, b, c \geqslant 2$, $d, e \geqslant 1$, $\kappa, \lambda \in \{0, 1\}$ and $\lambda \leqslant \kappa$. Here $\kappa = 0(1)$ means that there is no edge (is an edge) between e_{d+e} and e_{a+d+e} ($\lambda = 0(1)$ analogously). In Table 2 the values of d and e can be interchanged and for $\kappa = d = e = 1$, $\lambda = 0$ all values $b', c' \geqslant 2$ with $b' + c' = b + c$ (b, c in the table) are possible. Finally $i, j, k \geqslant 0$.

We shall examine the graph $R^{\kappa\lambda}_{abcde}$ more closely. Such a labelled weighted graph defines in an obvious way a lattice and a basis in this lattice (setting $\langle e_i, e_i\rangle = -2$ for all vertices e_i). The rank $rk(R^{\kappa\lambda}_{abcde})$ and discriminant $\operatorname{disc}(R^{\kappa\lambda}_{abcde})$ of the lattice defined by $R^{\kappa\lambda}_{abcde}$ are given by the following general formulas:

$$rk(R^{\kappa\lambda}_{abcde}) = a+b+c+d+e-1 = \mu,$$

$$\operatorname{disc}(R^{\kappa\lambda}_{abcde}) = (-1)^{\mu-1}.$$

$$\{[(1+\kappa+\lambda)c-1]\,(ab-a-b) - (1+\kappa+\lambda)ab - \kappa a(c+1)$$

$$-\lambda b(c+1) + (\kappa-\lambda)c\}\ de - [(c-1)ab-c(a+b)]\,(d+e).$$

Such a graph R also defines a Coxeter element C_R which is by definition the product of reflections corresponding to the vertices e_i,

$$C_R = s_{e_1} \circ \dots \circ s_{e_\mu}.$$

In the case that the graph is the Dynkin diagram of a distinguished basis, the Coxeter element C_R corresponds to the classical monodromy operator. Now by [2, Ch. V.6, Exercice 3] the characteristic polynomial $P_R(t)$ of C_R can be computed as follows

TABLE 2

Sing.	κ	λ	a	b	c	d	e
$J_{3,i}$	0	0	2	3	8+i	2	2
	0	0	2	3	8	2+i	2
$Z_{1,i}$	0	0	2	4	6+i	2	2
	0	0	2	4	6	2+i	2
$Q_{2,i}$	0	0	3	3	5+i	2	2
	0	0	3	3	5	2+i	2
$W_{1,i}$	0	0	2	5	5+i	2	2
	1	0	2	6	6	1+i	1
$W^{\#}_{1,i}$, $i>0$	0	0	2	5	5	2+i	2
$i=j+k-8$	1	0	2	2+j	2+k	1	1
	1	0	2	5	7	1+i	1
$S_{1,i}$	0	0	3	4	4+i	2	2
	1	0	3	5	5	1+i	1
$S^{\#}_{1,i}$, $i>0$	0	0	3	4	4	2+i	2
$i=j+k-6$	1	0	3	2+j	2+k	1	1
	1	0	3	4	6	1+i	1
$U_{1,i}$	1	0	4	2+j	2+k	1	1
$i=j+k-5$	1	0	4	4	5	1+i	1
$U_{1,1}$	1	1	4	5	5	1	1

Sing.	κ	λ	a	b	c	d	e
E_{18}	0	0	2	3	9	2	3
	0	0	2	3	8	3	3
E_{19}	0	0	2	3	10	2	3
	0	0	2	3	9	2	4
	0	0	2	3	8	3	4
E_{20}	0	0	2	3	11	2	3
	0	0	2	3	9	2	5
	0	0	2	3	8	3	5
Z_{17}	0	0	2	4	7	2	3
	0	0	2	4	6	3	3
Z_{18}	0	0	2	4	8	2	3
	0	0	2	4	7	2	4
	0	0	2	4	6	3	4
Z_{19}	0	0	2	4	9	2	3
	0	0	2	4	7	2	5
	0	0	2	4	6	3	5
Q_{16}	0	0	3	3	6	2	3
	0	0	3	3	5	3	3
Q_{17}	0	0	3	3	7	2	3
	0	0	3	3	6	2	4
	0	0	3	3	5	3	4
Q_{18}	0	0	3	3	8	2	3
	0	0	3	3	6	2	5
	0	0	3	3	5	3	5
W_{17}	0	0	2	5	6	2	3
	1	0	2	6	7	1	2
W_{18}	0	0	2	5	7	2	3
	1	0	2	7	7	1	2
	1	0	2	6	7	1	3
S_{16}	0	0	3	4	5	2	3
	1	0	3	5	6	1	2
S_{17}	0	0	3	4	6	2	3
	1	0	3	6	6	1	2
	1	0	3	5	6	1	3
U_{16}	1	0	4	5	5	1	2
	1	1	5	5	5	1	1

$$
\begin{aligned}
P_R(t) &= \det(t\,.\,1-C_R) \\
&= \begin{vmatrix}
1+t & -\langle e_1, e_2\rangle\, t & \dots & -\langle e_1, e_\mu\rangle\, t \\
-\langle e_2, e_1\rangle & 1+t & & \cdot \\
\cdot & & \cdot & \cdot \\
\cdot & & \cdot & \cdot \\
\cdot & & \cdot & \cdot \\
-\langle e_\mu, e_1\rangle & \dots & & 1+t
\end{vmatrix}
\end{aligned}
$$

In particular

$$P_R(1) = (-1)^\mu \operatorname{disc}(R)\,.$$

One can associate a directed graph R' to R as follows: Replace each edge between vertices e_i and e_j with $i < j$ by an arrow of the same type (dotted or not) pointing to e_j, and omit the numbering of the vertices. Then $P_R(t)$ depends only on R' and not on the special admissible numbering. Using the methods of [6], we have calculated $P_R(t)$ for $R = R^{\kappa\lambda}_{abcde}$ and obtained the following result. Let $I = \{a, b, c, d, e\}$ and for $J \subset I$ define ΣJ to be the formal expression

$$\sum_{j \in J} j\,.$$

Then the formal expression for $P_R(t)$ is

$$P_R(t) = (t-1)^{-5}\left(\sum_{\substack{J \subset I \\ \# J \leqslant 2}} \left(P_J(t)t^{\Sigma J} - P_J\left(\frac{1}{t}\right) t^{\mu+5-\Sigma J}\right)\right),$$

where

$$
\begin{aligned}
P_\varnothing &= (1+\kappa+\lambda)t^4 + 3t^3 - 6t^2 + 4t - 1\,, \\
P_{\{a\}} &= -(1+\kappa)t^4 - (1-\kappa+2\lambda)t^3 + (3-\kappa+2\lambda)t^2 - 3t + 1-\lambda\,, \\
P_{\{b\}} &= -(1+\lambda)t^4 - (1+\kappa)t^3 + (3+\lambda)t^2 - (3-\kappa+\lambda)t + 1-\kappa\,, \\
P_{\{c\}} &= -(\kappa+\lambda)t^4 - 2t^3 + (2+\kappa+\lambda)t^2 - (1+\kappa+\lambda)t\,, \\
P_{\{d\}} &= P_{\{e\}} = -(1+\kappa+\lambda)t^4\,, \\
P_{\{a,\,b\}} &= t^4 - (1-\kappa-\lambda)t^3 - (\kappa+\lambda)t^2 + 2t - (1-\kappa-\lambda)\,, \\
P_{\{a,\,c\}} &= \kappa t^4 + (1-\kappa+\lambda)t^3 - (1+\lambda)t^2 + (1+\kappa)t + \lambda\,, \\
P_{\{b,\,c\}} &= \lambda t^4 + t^3 - (1-\kappa+2\lambda)t^2 + (1-\kappa+2\lambda)t + \kappa\,,
\end{aligned}
$$

$$
\begin{aligned}
P_{\{a,\,d\}} &= P_{\{a,\,e\}} = (1+\kappa)t^4 - (1+\kappa-2\lambda)t^3 + (1+k-2\lambda)t^2 + \lambda\,,\\
P_{\{b,\,d\}} &= P_{\{b,\,e\}} = (1+\lambda)t^4 - (1-\kappa)t^3 + (1-\lambda)t^2 - (\kappa-\lambda)t + \kappa\,,\\
P_{\{c,\,d\}} &= P_{\{c,\,e\}} = (\kappa+\lambda)t^4 + (2-\kappa-\lambda)t^2 - (2-\kappa-\lambda)t + 1\,,\\
P_{\{d,\,e\}} &= (\kappa+\lambda)t^4 + t^3\,.
\end{aligned}
$$

Now given the characteristic polynomial of the classical monodromy operator of a bimodular singularity, one can compute the values of $\kappa, \lambda, a, b, c, d, e$ for which the polynomial above coincides with it. In this way one gets

SUPPLEMENT TO THEOREM 4.1. *Table 2 (the remarks after Theorem 4.1 taken into account) contains for each bimodular singularity all possible values* $\kappa, \lambda, a, b, c, d, e$ *such that the graph* $R^{\kappa\lambda}_{abcde}$ *is a Dynkin diagram with respect to a distinguished basis of the singularity.*

The graph S_{pqr} is related to the graph $R^{\kappa\lambda}_{abcde}$ in the following way. The group

$$Z^* = Z_\mu \rtimes (\mathbf{Z}/2\mathbf{Z})^\mu$$

acts also on the set of all labelled graphs weighted by ± 1 with μ vertices. We denote equivalence under Z^* by $\sim$. Then

$$
\begin{aligned}
&R^{00}_{abc1e} \sim R^{00}_{a,\,b,\,c+1,\,1,\,e-1} && (e \geqslant 2)\\
&R^{00}_{abc11} \sim S_{a,\,b,\,c+1} &&\\
&R^{00}_{abcd1} \sim R^{00}_{a,\,b,\,c+1,\,d-1,\,1} && (d \geqslant 2)
\end{aligned}
$$

(*Proof:* $\beta_3, \beta_4, \ldots, \beta_\mu, \beta_\mu, \beta_{\mu-1}, \gamma_{\mu-2}$).

Therefore Theorem 4.1 and the supplement above imply in particular that none of the bimodular singularities has a distinguished basis with a Dynkin diagram of type S_{pqr}.

A closer study of Table 2 yields the following observation, with which we want to conclude. Let $R^{\kappa\lambda}_{abcde}$ be a graph of a singularity X of Table 2. Substract 1 from one of the following parameters:

c, d, e	for the E/J-, Z-, Q- series
b, c, d, e	for the W-, S- series
a, b, c, d, e	for the U- series

such that the new parameters $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{e}$ still satisfy $\tilde{a}, \tilde{b}, \tilde{c} \geqslant 2$, $\tilde{d}, \tilde{e} \geqslant 1$. Then either $R^{\kappa\lambda}_{\tilde{a}\tilde{b}\tilde{c}\tilde{d}\tilde{e}}$ is again a graph of Table 2, say of the singularity Y, and we relate X and Y by an arrow $X \to Y$. Or it is equivalent under Z^* to a graph of the form

S_{pqr} which does not correspond to a distinguished basis of any unimodular singularity. So the graphs of the bimodular singularities cannot be simplified by the action of Z^* to a graph S_{pqr}, but the graphs immediately "below" them can. On the other hand the relations one gets by the arrows are exactly the adjacency relations of Laufer [15] between bimodular singularities with the difference of the Milnor numbers being equal to 1.

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