2. Geometric Bases of the Milnor Lattice

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2. Geometric Bases of the Milnor Lattice

There are certain classes of bases for the Milnor lattice, which are distinguished by the geometry of the singularity. We shall recall the definition of these bases (see [12] for more information). For that purpose we shall consider the semiuniversal deformation of the singularity.

Let $X_0 = f^{-1}(0)$ be the complex analytic hypersurface defined by f. Let F: $X \to S$ be a suitable representative of the semi-universal deformation of the germ $(X_0, 0)$. Denote by $D \subset S$ the corresponding discriminant, i.e. the image of the critical locus of F. Put $S' = S - D$, $X' = F^{-1}(S')$ and $F' = F|_{X'}$. Then $F': X' \to S'$ is a C^∞-fiber bundle, where each fiber is diffeomorphic to the Milnor fiber.

Choose a generic complex line in the affine space containing S, which intersects S in a complex disc Δ. Then Δ intersects the discriminant D in μ points $c_1, ..., c_\mu$, which lie in the interior of Δ. Choose a basepoint s_0 on the boundary of Δ. Let X_{s_0} denote the fiber of F over s_0. Then $H_{n-1}(X_{s_0}, \mathbf{Z})$ is isomorphic to L and will also be denoted by L. We shall construct bases of L. Each path ϕ_i in $\Delta' = \Delta - \{c_1, ..., c_\mu\}$ from c_i to s_0 determines an element of L as follows. The fiber over c_i has only one singular point which is an ordinary double point. Near this point, the fibers are given locally by an equation

$$z_1^2 + ... + z_n^2 = r^2 .$$

The real sphere

$$S^{n-1} = \{(x_1, ..., x_n) \mid x_1^2 + ... + x_n^2 = r^2\}, \ z_i = x_i + iy_i ,$$

represents after the choice of an orientation a homology class in a fiber over a point of ϕ_i near c_i. Transport along the path ϕ_i gives an element $e \in L$ satisfying $\langle e, e\rangle = -2$. Such an element is called *vanishing cycle*. Let Λ^* denote the set of vanishing cycles. Choosing a path ϕ_i for each c_i yields a system of μ vanishing cycles.

In order to get a basis of vanishing cycles, there are several possible restrictions on the choice of paths. In order to define these restrictions and the corresponding classes of bases, we need the notion of a *simple loop* corresponding to a path ϕ_i from c_i to s_0. This is the element of $\pi_1(\Delta', s_0)$ represented by the loop τ_i going from s_0 to a point sufficiently near to c_i along the path ϕ_i, going once around c_i in the positive direction (counterclockwise) and returning to s_0 along the path ϕ_i. This loop induces an automorphism of L, the Picard-Lefschetz-transformation corresponding to the path ϕ_i. It is given by the reflection s_{e_i} on the orthogonal complement of the vanishing cycle e_i corresponding to ϕ_i, i.e. s_{e_i} is defined by

$$s_{e_i}(x) = x + \langle x, e_i\rangle e_i \quad \text{for } x \in L .$$

1) *Distinguished Bases*

Choose the paths ϕ_i non-selfintersecting, let any two have only s_0 as a common point and number the paths in the order in which they arrive at s_0, counted clockwise beginning from the boundary of Δ. Then the corresponding system of vanishing cycles forms a basis, and a basis of L obtained in this way is called a *distinguished basis.* Let $\mathscr{B}^*$ denote the set of all distinguished bases of L.

On $\mathscr{B}^*$ there is an operation of the braid group Z_μ in μ strings, where a generator α_i operates as follows: Let $B = \{e_1, ..., e_\mu\} \in \mathscr{B}^*$ be a distinguished basis defined by the system of paths $\{\phi_1, ..., \phi_\mu\}$. The operation α_i is induced by the following elementary operation on the level of paths (cf. Fig. 1):

$$(\phi_1, ..., \phi_\mu) \to (\phi_1, ..., \phi_{i-1}, \phi_{i+1}\tau_i, \phi_i, \phi_{i+2}, ..., \phi_\mu) .$$

The operation α_i is then given by

$$(e_1, ..., e_\mu) \to (e_1, ..., e_{i-1}, s_{e_i}(e_{i+1}), e_i, e_{i+2}, ..., e_\mu) .$$

The inverse operation α_i^{-1} is also denoted by β_{i+1}.

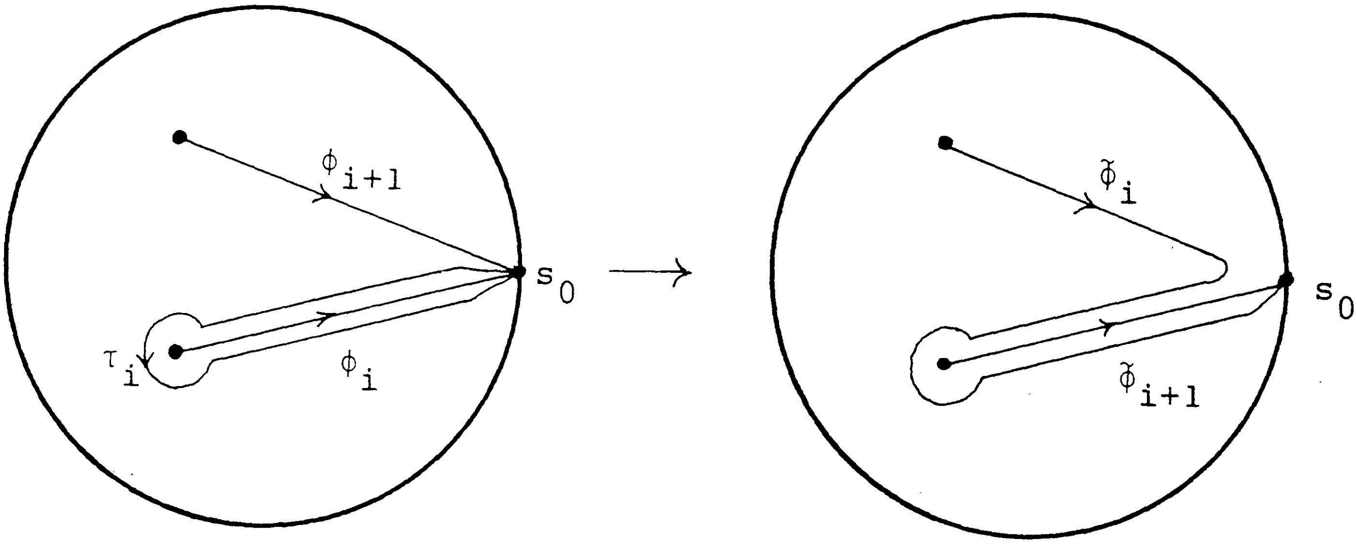

FIGURE 1

A distinguished basis does not only depend on the choice of paths, but also on the choice of orientation of the cycles. The change of orientation of a cycle e_i corresponds to an operation γ_i given by $\gamma_i(e_j) = e_j$ for $j \neq i$ and $\gamma_i(e_i) = -e_i$.

Therefore there is in addition an operation of $(\mathbf{Z}/2\mathbf{Z})^\mu$ on $\mathscr{B}^*$. Together one has an operation of the semidirect product

$$Z^* = Z_\mu \rtimes (\mathbf{Z}/2\mathbf{Z})^\mu$$

on $\mathscr{B}^*$ (both factors considered as subgroups of the symmetric group of $\mathscr{B}^*$).

PROPOSITION 2.1. *The operation of Z^* on $\mathscr{B}^*$ is transitive.*

For a proof see [12].

PROPOSITION 2.2. *If $\mathscr{B}^*$ contains a basis $B = \{e_1, \ldots, e_\mu\}$ with $\langle e_i, e_j \rangle \in \{0, 1, -1\}$ for $i \neq j$, then already Z_μ operates transitively on $\mathscr{B}^*$.*

Proof. It suffices to show that any basis $\tilde{B} \in \mathscr{B}^*$ can be transformed into B by Z_μ. By the previous proposition there exists an element of Z^* which transforms $\tilde{B}$ to B. We show that for each i there exists an element $\sigma_i \in Z_\mu$ such that $\gamma_i(B) = \sigma_i(B)$. If $\langle e_j, e_{j+1} \rangle = \varepsilon$, $\varepsilon = \pm 1$, then $\alpha_j^{12} = id$ and

$$\alpha_j^{3\varepsilon}(B) = \gamma_{j+1}(B), \ \alpha_j^{-3\varepsilon}(B) = \gamma_j(B) .$$

Now let k be the smallest integer such that $\langle e_i, e_{i+k} \rangle \neq 0$, $k \neq 0$. We consider the case $k > 0$, the case $k < 0$ is analogous. If $k = 1$, we apply the previous remark. Otherwise the transformation $(\alpha_{i+k-2} \circ \ldots \circ \alpha_i)(B)$ interchanges e_i and e_{i+k-1} and leaves all other basis elements fixed. Hence one can now apply the above remark. This proves the proposition.

Let $B = \{e_1, \ldots, e_\mu\}$ be a distinguished basis. Then

$$h = s_{e_1} \circ s_{e_2} \circ \ldots \circ s_{e_\mu}$$

is the *classical monodromy operator* of the singularity.

2) *Weakly Distinguished Bases*

We now impose the only condition on the system of paths that the corresponding simple loops generate $\pi_1(\Delta', s_0)$. Then it can be shown that the corresponding system of vanishing cycles forms again a basis, and a basis obtained in this way is called a *weakly distinguished basis*. Let $\mathscr{B}^0$ denote the set of all weakly distinguished bases.

Since the numbering does not play a role for a weakly distinguished basis, we have on $\mathscr{B}^0$ also an operation of the symmetric group $\mathscr{S}_\mu$ of degree μ. Let Z^0 be the group generated by Z^* and $\mathscr{S}_\mu$. There are some special elements in Z^0 defined as follows: Let $B = \{e_1, ..., e_\mu\}$ be a weakly distinguished basis defined by a system of paths $\{\phi_1, ..., \phi_\mu\}$. Let $\{\tau_i\}$ be the corresponding system of simple loops. For $i \neq j$ we define a transformation $\alpha_i(j)$ on the different levels as follows:

$$(\phi_1, ..., \phi_\mu) \to (\phi_1, ..., \phi_{j-1}, \phi_j\tau_i, \phi_{j+1}, ..., \phi_\mu),$$

$$(\tau_1, ..., \tau_\mu) \to (\tau_1, ..., \tau_{j-1}, \tau_i^{-1}\tau_j\tau_i, \tau_{j+1}, ..., \tau_\mu),$$

$$(e_1, ..., e_\mu) \to (e_1, ..., e_{j-1}, s_{e_i}(e_j), e_{j+1}, ..., e_\mu).$$

The inverse transformation is denoted by $\beta_i(j)$. In the case $n \equiv 3(4)$ it coincides with $\alpha_i(j)$ on the homology level. Now Z^0 is generated by the transformations $\alpha_i(j)$, $\mathscr{S}_\mu$ and change of orientation operations.

CONJECTURE 2.3 (Gusein-Zade). *The operation of* Z^0 *on* $\mathscr{B}^0$ *is transitive.*

This conjecture can be reduced to a problem in pure combinatorial group theory, see [12]. It is not known to the author, whether the conjecture is true.

The *monodromy group* Γ of the singularity f is the image of $\pi_1(\Delta', s_0)$ under the natural representation

$$\rho : \pi_1(\Delta', s_0) \to \mathrm{Aut}(L).$$

It is generated by the reflections corresponding to the elements of a weakly distinguished basis.

The matrix of the bilinear form on L with respect to a basis B of vanishing cycles is described in the usual way by a graph with (possibly multiple) edges weighted by $+1$ or -1, where we indicate negative weight by a dotted line. This graph is called the *Dynkin diagram* with respect to B.

3. Milnor Lattices and Weakly Distinguished Bases of Some Special Singularities

We shall consider the Milnor lattices of some specific singularities, namely the singularities of Arnold's lists and the minimally elliptic hypersurface singularities. By the singularities of Arnold's lists we mean the singularities, for which Arnold has given normal forms in [1], i.e. the singularities of the series A, D, J, E, X, Y, Z, W, T, Q, S, U and V. Most of these series contain singularities with arbitrary number of moduli. The minimally elliptic hypersurface singularities can be defined as follows (cf. [5]): They are the singularities f: