3. Milnor Lattices and Weakly Distinguished Bases of Some Special Singularities

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **29 (1983)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **21.07.2024**

the group generated by Z^* and $\mathcal{S}_\mu$. There are some special elements in Z^0 defined as follows: Let $B = \{e_1, ..., e_\mu\}$ be a weakly distinguished basis defined by a system of paths $\{\phi_1, ..., \phi_\mu\}$. Let $\{\tau_i\}$ be the corresponding system of simple loops. For $i \neq j$ we define a transformation $\alpha_i(j)$ on the different levels as follows:

$$(\phi_1, ..., \phi_\mu) \to (\phi_1, ..., \phi_{j-1}, \phi_j\tau_i, \phi_{j+1}, ..., \phi_\mu),$$
$$(\tau_1, ..., \tau_\mu) \to (\tau_1, ..., \tau_{j-1}, \tau_i^{-1}\tau_j\tau_i, \tau_{j+1}, ..., \tau_\mu),$$
$$(e_1, ..., e_\mu) \to (e_1, ..., e_{j-1}, s_{e_i}(e_j), e_{j+1}, ..., e_\mu).$$

The inverse transformation is denoted by $\beta_i(j)$. In the case $n \equiv 3(4)$ it coincides with $\alpha_i(j)$ on the homology level. Now Z^0 is generated by the transformations $\alpha_i(j)$, $\mathcal{S}_\mu$ and change of orientation operations.

CONJECTURE 2.3 (Gusein-Zade). *The operation of* Z^0 *on* $\mathcal{B}^0$ *is transitive.*

This conjecture can be reduced to a problem in pure combinatorial group theory, see [12]. It is not known to the author, whether the conjecture is true.

The *monodromy group* Γ of the singularity f is the image of $\pi_1(\Delta', s_0)$ under the natural representation

$$\rho: \pi_1(\Delta', s_0) \to \mathrm{Aut}(L).$$

It is generated by the reflections corresponding to the elements of a weakly distinguished basis.

The matrix of the bilinear form on L with respect to a basis B of vanishing cycles is described in the usual way by a graph with (possibly multiple) edges weighted by $+1$ or -1, where we indicate negative weight by a dotted line. This graph is called the *Dynkin diagram* with respect to B.

3. Milnor Lattices and Weakly Distinguished Bases of Some Special Singularities

We shall consider the Milnor lattices of some specific singularities, namely the singularities of Arnold's lists and the minimally elliptic hypersurface singularities. By the singularities of Arnold's lists we mean the singularities, for which Arnold has given normal forms in [1], i.e. the singularities of the series A, D, J, E, X, Y, Z, W, T, Q, S, U and V. Most of these series contain singularities with arbitrary number of moduli. The minimally elliptic hypersurface singularities can be defined as follows (cf. [5]): They are the singularities f:

$(\mathbf{C}^3, 0) \to (\mathbf{C}, 0)$ with $\mu_0 + \mu_+ = 2$. They have been classified by Laufer (cf. [14]). Both classes of singularities contain in particular all uni- and bimodular singularities.

By the methods of [11] one can show that all the above singularities have a distinguished basis $B = \{e_1, ..., e_\mu\}$ satisfying $\langle e_i, e_j\rangle \in \{0, 1, -1\}$ for $i \neq j$, i.e. satisfy the conditions of Prop. 2.2. Using the operation of Z^0, we look for other elements of the sets $\mathscr{B}^0$, which reveal more of the structure of the Milnor lattice. In [7] we have listed weakly distinguished bases for the singularities of Arnold's lists except the series V, which give rise to certain orthogonal splittings of the corresponding Milnor lattices. From these results we also derived that the monodromy groups of almost all of these singularities can be characterized arithmetically, which is even true for a much larger class of singularities (cf. [8, 9]). The orthogonal splittings enable one to compute in an easy way the discriminant quadratic forms of the corresponding Milnor lattices. In particular one gets the following result. Let $\lambda(G_L)$ denote the minimal number of generators of G_L.

THEOREM 3.1. *The following is true for all singularities of Arnold's lists:*

(i) $\mu_0 \leqslant 2$, $\mu_- \geqslant 5(\mu_0+\mu_+) - 4$.

(ii) *The number* $\mu_0 + \mu_+$ *grows proportional to the number of moduli within each series.*

(iii) $\lambda(G_L) \leqslant 3$.

For the minimally elliptic hypersurface singularities one can derive the following result. We first define a graph-theoretical invariant. Let H be a graph. For a vertex $v \in H$, the degree of v, $\deg v$, is the number of edges incident with v. Let $z(H)$ be the number of cycles of H of the form $v_0, v_1, ..., v_r = v_0$, where there exists a number k, $1 \leqslant k < r$, with $\deg v_i \geqslant 3$ for $1 \leqslant i \leqslant k$ and $\deg v_i = 2$ otherwise. Define

$$\sigma(H) = \sum_{\substack{v \in H \\ \deg v \geqslant 3}} (\deg v - 2) + z(H)\,.$$

THEOREM 3.2.

(i) *Let* f *be a minimally elliptic hypersurface singularity with* $\mu_+ = 2$ *(hence* $\mu_0 = 0$*). Then there exists a weakly distinguished basis* $B = \{e_1, ..., e_\mu\}$ *of* f *satisfying the following properties:*

a) $\langle e_1, e_2\rangle = 1$,

$\langle e_1, e_i\rangle = 0$, $\langle e_2, e_i\rangle = \langle e_3, e_i\rangle$ *for* $3 \leqslant i \leqslant \mu$.

b) *For* $i, j \in \{3, ..., \mu\}$, $i \neq j$, $\langle e_i, e_j \rangle \in \{0, 1\}$,
(*The matrix* $(-\langle e_i, e_j \rangle)_{3 \leqslant i, j \leqslant \mu}$ *is therefore an indecomposable symmetric Cartanmatrix of negative type in the sense of* [13]).

c) *Let* H *denote the subgraph of the Dynkin diagram with respect to* $\{e_3, ..., e_\mu\}$. *Then*

$$1 \leqslant \sigma(H) \leqslant 4 .$$

(ii) *For all minimally elliptic hypersurface singularities*

$$\lambda(G_L) \leqslant 4 .$$

More information about the Milnor lattices of these singularities will be given in a forthcoming paper. Dynkin diagrams corresponding to weakly distinguished bases satisfying (i) are given for the unimodular singularities in [10] (here $\sigma(H) = 1$) and for the bimodular in [7].

THEOREM 3.3. *For each bimodular singularity* $\mu_0 = 0, \mu_+ = 2, \lambda(G_L) \leqslant 3$. *Moreover there exists a weakly distinguished basis satisfying Th. 3.2* (i) *with* $\sigma(H) = 2$.

Example. Consider the following two bimodular families of singularities:

$$E_{18}: x^3 + y^{10} + z^2 + a_0 x y^7 + a_1 x y^8$$
$$Q_{18}: x^3 + y z^2 + y^8 + a_0 x y^6 + a_1 x y^7 .$$

No member of the class E_{18} is topologically equivalent to a member of the class Q_{18}, since the resolution graphs are different and by Neumann's result [16] the corresponding links are not diffeomorphic. This implies in particular that the corresponding Milnor fibers are not diffeomorphic. But the singularities of both families have the same discriminant 3 and the same signature $(\mu_0, \mu_+, \mu_-) = (0, 2, 16)$. By a result in the theory of quadratic forms [7, Satz 2.2], there is up to isomorphism only one lattice with these invariants. So both singularity classes have the same Milnor lattice L, and an explicit description of L is e.g. given by

$$L = E_6 \perp E_8 \perp U \perp U .$$

Moreover the sets $\mathscr{B}^0$ coincide in both cases, implying also that the monodromy groups Γ are the same. Dynkin diagrams with respect to weakly distinguished bases satisfying the conditions of Theorem 3.2 are given by the graph of Fig. 2, where the quintuples (a, b, c, d, e) are listed in Table 1. These are also the only possibilities of a graph of the form of Fig. 2 to be a graph of the above lattice L. The graph of Fig. 2 satisfies $\sigma = 2$.

However, the sets $\mathscr{B}^*$ are different for the two classes of singularities, because the classical monodromy operators have different orders, namely 30 for E_{18} and 48 for Q_{18}.

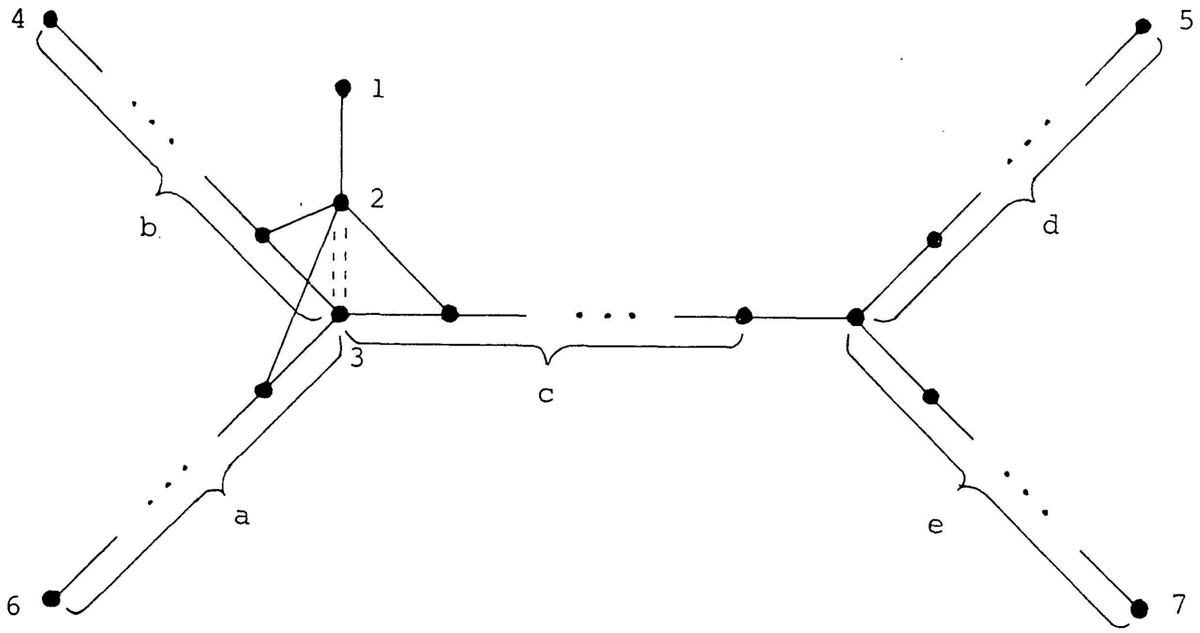


FIGURE 2

TABLE 1

a	b	c	d	e
2	3	9	2	3
2	3	8	3	3
2	5	6	3	3
3	3	5	3	5

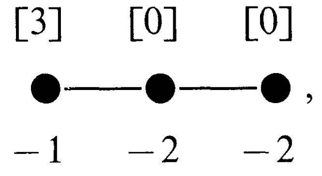
Remark. There are also examples of singularities with different numbers of moduli which have isomorphic L, Γ and $\mathscr{B}^0$. Moreover J. Wahl has informed me that H. Laufer has found an example of two singularities of different topological type which even have the same resolution graph and hence diffeomorphic links, isomorphic Milnor lattices and by [8] isomorphic monodromy groups. These are the singularities given by

$$z^3 + x^4 + y^{36}$$

and

$$z^2 + y(x^{12} + y^{18}).$$

The resolution graph is in both cases



where the number in brackets denotes the genus, the other the selfintersection number of the corresponding cycle. Here $(\mu_0, \mu_+, \mu_-) = (6, 42, 162)$. However, the orders of the classical monodromy operators are 36 resp. 38.

4. Distinguished Bases for the Bimodular Singularities

We have seen in the last section that there are bimodular singularities which have the same Dynkin diagrams with respect to weakly distinguished bases, but not with respect to distinguished bases. We now turn our attention to the sets $\mathscr{B}^*$ for these singularities. Let us first look at the unimodular case. All exceptional unimodular singularities have a weakly distinguished basis with a Dynkin

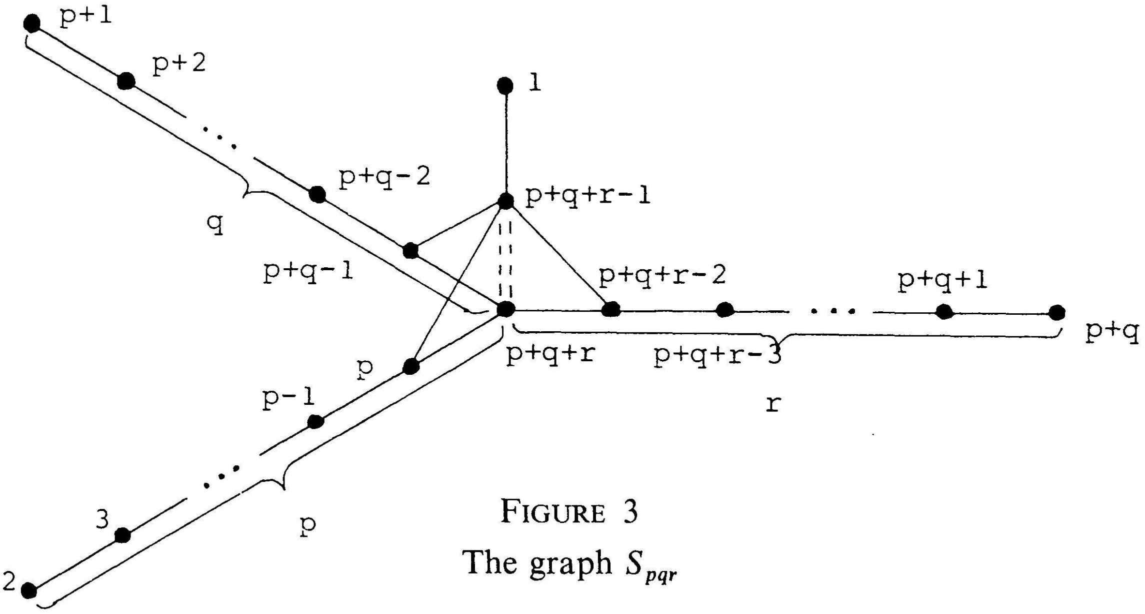


Figure 3
The graph S_{pqr}