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THE ACTION OF THE MAPPING CLASS GROUP ON CURVES IN SURFACES

par Robert C. PENNER

In this paper, we will discuss the solution to a problem originally suggested by Max Dehn in 1922 [D1] and more recently posed by William Thurston in [T3, problem number 20]. The problem is to compute the action of homeomorphisms of a surface on one-manifolds embedded in the surface. This computation has several applications to Riemann surface theory, dynamics of surface automorphisms, and low-dimensional topology. To give a precise statement to this problem requires a concise way to specify both homeomorphisms of surfaces and one-submanifolds of surfaces; we will discuss this background material.

This paper is a survey of some of the results in my thesis [P]; I would like to thank Dave Gabai for introducing me to some of this material and for sharing with me his initial work and insights on the main problem. Thanks also to James Munkres for his suggestions and encouragement.

Let F_g denote the g-holed torus and let H_g^+ denote the topological group of orientation-preserving homeomorphisms of F_g (with the compact-open topology). The *mapping class group* of F_g, which we will denote $MC(F_g)$, is defined to be the group H_g^+ modulo isotopy. By definition, this is the same as the group of path components of the space H_g^+. Moreover, Nielsen [N] shows that $MC(F_g)$ may be identified with the group of (orientation-preserving) outer automorphisms of the fundamental group of F_g. The mapping class groups are central objects of study in Riemann surface theory as well as in two- and three-dimensional topology. For instance, a useful technique is cutting and regluing a three-manifold along an embedded surface. The homeomorphism type of the resulting three-manifold depends only on the isotopy class of the gluing map.

I do not know who first studied the groups $MC(F_g)$, but they have been actively researched since the beginning of this century. M. Dehn [D2] was the first to give a finite set of generators for $MC(F_g)$ of a certain geometrical type which are now called *Dehn twists*. If c is a simple closed curve embedded in F_g, then the right and left Dehn twists along c, denoted $\tau_c^{\pm 1}: F_g \to F_g$, are defined as follows: cut F_g along c, twist once around to the right or left and reglue. Thus, if c and d are as shown in Figure 1, then the curves $\tau_c^{\pm 1}d$ are as pictured. The direction (right or left) of a Dehn twist is independent of an orientation on c and depends only on the orientation of the surface F_g.

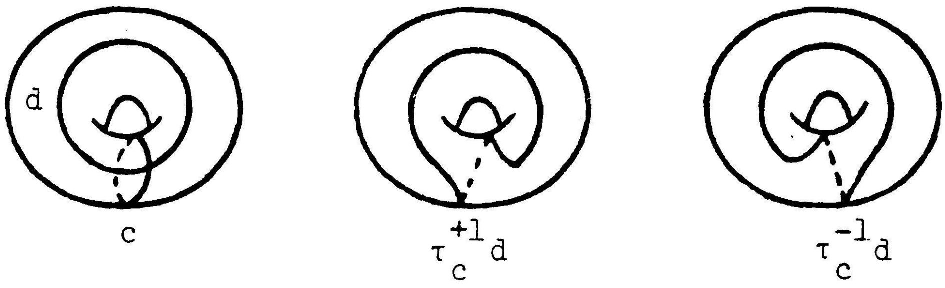

FIGURE 1

In 1938, Dehn [D2] described a finite collection of Dehn twist generators for $MC(F_g)$, and in 1964-66, R. Lickorish [L] independently refined Dehn's original set to a more useful collection of $3g - 1$ curves along which to perform Dehn twists. For later use, we will record Lickorish's result as a theorem.

THEOREM [Lickorish]. *For* $g \geqslant 2$, $MC(F_g)$ *is generated by the Dehn twists along the curves pictured in Figure 2.*

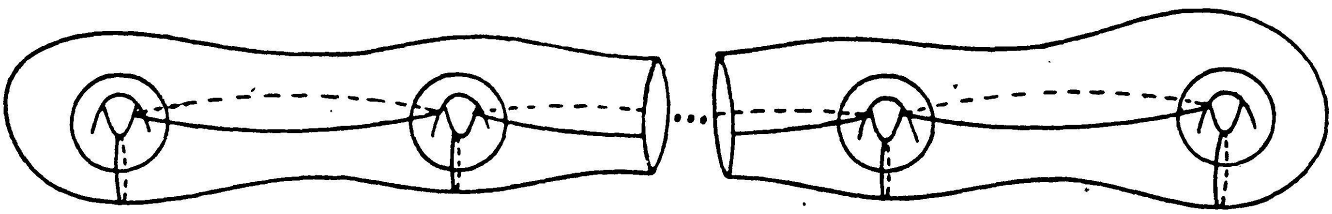

FIGURE 2

Using the result of Nielsen stated earlier, it is easy to see that $MC(F_1)$ is isomorphic to $Sl_2(\mathbf{Z})$, generated by the Dehn twists along the curves c and d pictured in Figure 1. For a closed surface of genus two, J. Birman and M. Hilden [BH] have given a complete set of relations amongst the Lickorish generators. For closed surfaces of arbitrary genus, A. Hatcher and W. Thurston [HT] have given an algorithm for constructing a complete set of relations for $MC(F_g)$, but their results are quite complicated. More

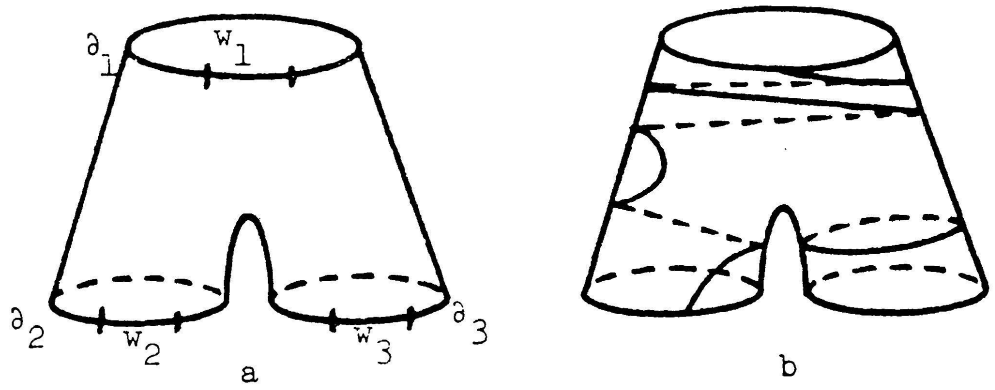

FIGURE 3

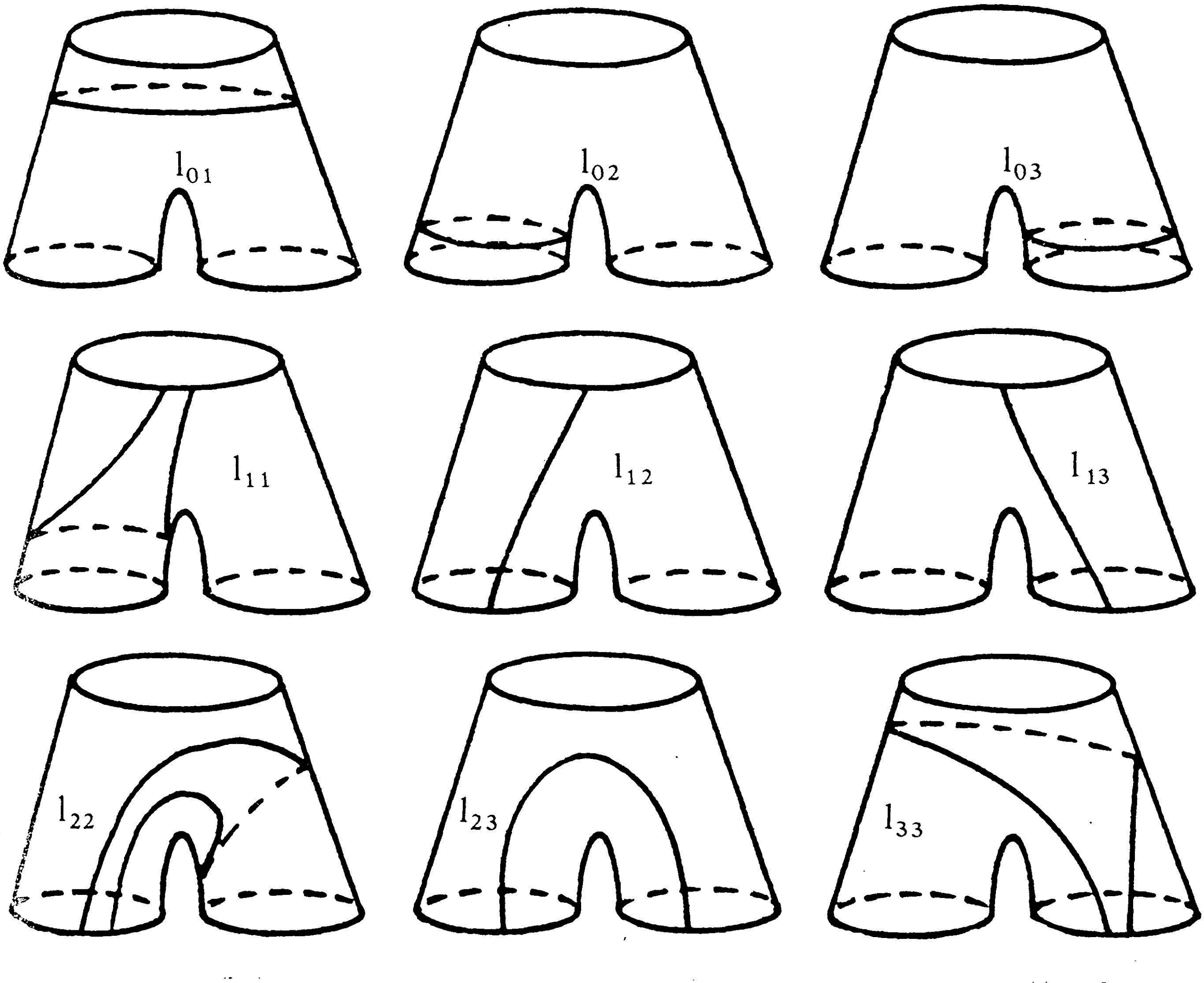

FIGURE 4

recently, B. Wajnryb [W] has carried through the beautiful techniques introduced in [HT] and given a presentation for $MC(F_g)$, $g \geqslant 2$.

There is a great deal more known about structural properties of the groups $MC(F_g)$ (see [B]), but this introduction to generators and relations should suffice for our present purposes.

We will explicitly compute a certain action of the groups $MC(F_g)$: the natural action of H_g^+ on (non-oriented) curves embedded in F_g descends to an action of $MC(F_g)$ on isotopy classes of curves in F_g. Thus, given $[\phi] \in MC(F_g)$ and an isotopy class $[c]$ of curves on F_g, we will compute the isotopy class $[\phi(c)]$. We require some concise way to describe an isotopy class of curves in F_g. In fact, we will describe a one-to-one correspondence between such isotopy classes and a subset of Z^N, for some big N. Such a one-to-one correspondence will be termed a *parametrization.*

In 1922, Dehn [D1] described such a parametrization in a Breslau lecture. This work was not published and remained generally unknown. In 1976, W. Thurston [T2] independently rediscovered and generalized Dehn's parametrization. We call this parametrization the Dehn-Thurston parametrization and will presently describe it.

Though we are primarily interested in curves embedded in g-holed tori F_g, more generally we will be led to consider a one-manifold c properly embedded in an oriented surface F perhaps with boundary. We choose once and for all an arc, called a *window*, in each boundary component of F. We require that ∂c is contained in the windows, that no closed component of c bound a disc in F, and that no arc component of c can be isotoped into ∂F. Define a *multiple arc* in F to be an isotopy (fixing the boundary pointwise except in the windows) class of such one-submanifolds, and denote the collection of multiple arcs in F by $\mathscr{S}'(F)$. The Dehn-Thurston theorem gives a parametrization of $\mathscr{S}'(F)$, provided the Euler characteristic of F is negative. Exposé 4 of [FLP] contains a proof of the result we describe below.

We will first consider $\mathscr{S}'(F)$ for a particularly simple surface F. A *pair of pants* P is a disc-minus-two-discs with the boundary components denoted ∂_i and the windows w_i, $i = 1, 2, 3$, as indicated in Figure 3*a*. Let Δ_i be a fixed neighborhood of ∂_i in P, and consider the nine examples of one-manifolds l_{ij}, $0 \leqslant i \leqslant j \leqslant 3$, illustrated in Figure 4. We assume that $l_{0j} \subset \Delta_j$, $j = 1, 2, 3$. In case $[c] \in \mathscr{S}'(P)$ is represented by a connected one-manifold, one can show that c is isotopic to one of the models l_{ij}. Moreover, if c is represented by an arc properly embedded in P, then c is isotopic (fixing ∂P pointwise) to an arc c' which agrees with some l_{ij}, $i \neq 0$, on $P \backslash (U_k \Delta_k)$ and twists to the right or left some number of times in each of Δ_i and Δ_j.

One may thus parametrize connected $[c] \in \mathscr{S}'(P)$ as follows. Let m_i denote the number of times c (or c') intersects ∂_i, $i = 1, 2, 3$. Furthermore, define integers t_i, $i = 1, 2, 3$, by taking $|t_i|$ to be the number of times c' twists in Δ_i with the sign of t_i positive if c' twists to the right in Δ_i and negative if c' twists to the left in Δ_i. Moreover, if c is isotopic to l_{0j} so that $m_j = 0$, then we choose to call the twisting positive. Thus, the examples in Figure 3*b* have parameter values $t_3 = 1$, $t_1 = t_2 = m_1 = m_2 = m_3 = 0$ and $m_1 = m_2 = t_2 = 1$, $t_1 = -2$, $m_3 = t_3 = 0$.

This gives a parametrization of connected $[c] \in \mathscr{S}'(P)$ by a six-tuple $(m_i) \times (t_i) \in (\mathbf{Z}^+)^3 \times \mathbf{Z}^3$ of integers. To extend this to a parametrization of arbitrary (possibly disconnected) $[c] \in \mathscr{S}'(P)$, one simply lets $\{c_n\}$ denote the components of c and defines $m_i([c]) = \Sigma_n m_i([c_n])$ and $t_i([c]) = \Sigma_n t_i([c_n])$. Notice that if c_1 and c_2 are components of c and $m_i([c_1]) \neq 0 \neq m_i([c_2])$ for some $i = 1, 2, 3$, then $t_i([c_1])$ and $t_i([c_2])$ have the same sign since c_1 and c_2 are disjointly embedded. Similarly, if some component of c is isotopic to l_{0j}, $j = 1, 2, 3$, then $m_j([c]) = 0$.

For disconnected $[c] \in \mathscr{S}'(P)$, the parameters $m_i([c])$ and $t_i([c])$ have the following geometrical interpretation. We say d is an *ε-translate* of l_{ij} if there is a neighborhood of l_{ij} identified with the unit normal bundle $l_{ij} \times [-1, 1]$ so that d corresponds to $l_{ij} \times \varepsilon$. Given $[c] \in \mathscr{S}'(P)$, we choose c' representing $[c]$ so that c' agrees with a collection of ε-translates of the l_{ij} on $P \backslash (U_k \Delta_k)$. $m_i([c])$ is simply the number of times c' intersects ∂_i, and $t_i([c])$ is the total twisting of c' in Δ_i. Notice that for any $[c] \in \mathscr{S}'(P)$, $\Sigma_i m_i([c])$ is even since representatives of $[c]$ are properly embedded.

To parametrize $[c] \in \mathscr{S}'(F_g)$, $g \geqslant 2$, we choose a decomposition, called a *pants decomposition*, of F_g into pairs of pants: we choose a collection of curves with windows $\{(K_i, u_i)\}$ so that each component of $F_g \backslash \cup \{K_i\}$ is the interior of a pair of pants. (For Euler characteristic reasons, there are $3g - 3$ pants curves K_i in a pants decomposition of F_g.) Some examples of pants decompositions of F_2 are shown in Figure 5. Note that we do not require the closure of a component of $F_g \backslash \cup \{K_i\}$ to be a pair of pants; see Figures 5*b* and *c*.

Let A_i be an annular neighborhood of K_i, and identify once and for all each component of $F_g \backslash \cup \{\text{Interior } A_i\}$ with the pair of pants P. Given some $[c] \in \mathscr{S}'(F_g)$, we compute the corresponding parameter values as follows. Isotope c so that it intersects each K_i a minimal number of times, and let m_i be this intersection number. Now isotope c so that it intersects K_i in the window u_i (if at all) and so that (relative to our identifications) it coincides with ε-translates of l_{ij} in each component of $F_g \backslash \cup \{\text{Interior } A_i\}$; let

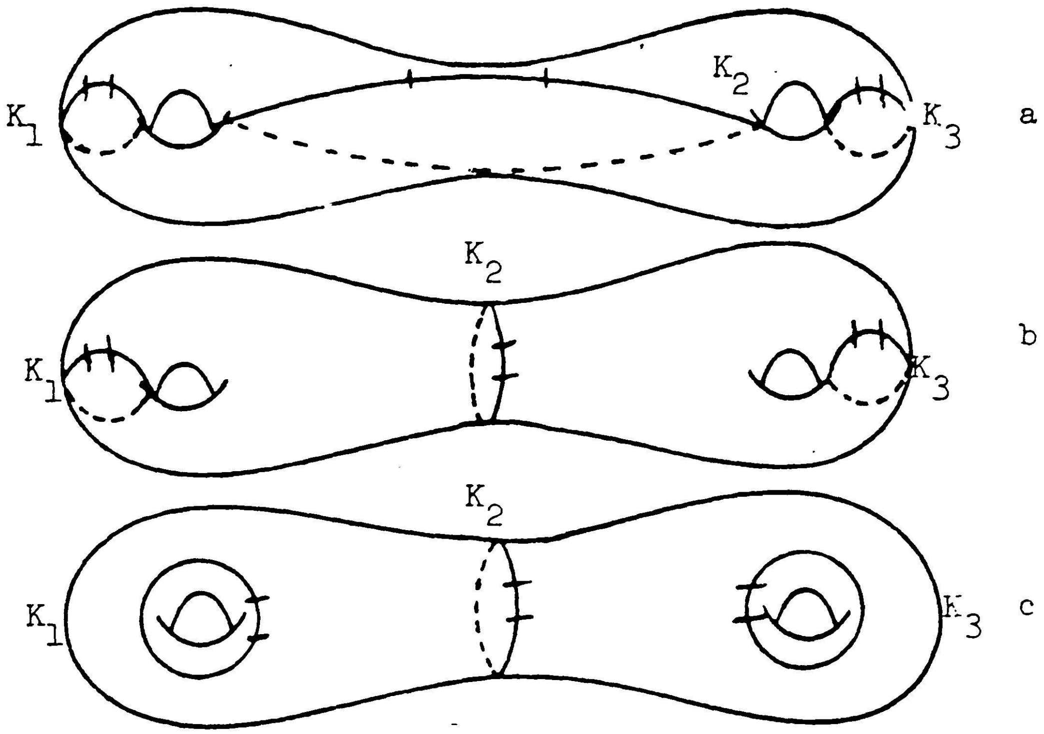

FIGURE 5

t_i be the number of times c twists in the annulus A_i with the conventions as before. The $(6g-6)$-tuple of integers $(m_i) \times (t_i) \in (\mathbf{Z}^+)^{3g-3} \times \mathbf{Z}^{3g-3}$ is the Dehn-Thurston parameter value for $[c]$.

THEOREM [Dehn-Thurston]. *$\mathcal{S}'(F_g)$ is parametrized by a subset of $(\mathbf{Z}^+)^{3g-3} \times \mathbf{Z}^{3g-3}$. A point $(m_i) \times (t_i) \in (\mathbf{Z}^+)^{3g-3} \times \mathbf{Z}^{3g-3}$ corresponds to a multiple arc if and only if the following conditions are satisfied.*

a) *If $m_i = 0$ then $t_i \geqslant 0$.*

b) *If K_i, K_j, and K_k are pants curves that bound an embedded pair of pants in F_g, then $m_i + m_j + m_k$ is even.*

c) *If K_i is a pants curve that bounds an embedded torus-minus-a-disc, in F_g, then m_i is even.*

Restriction a) is simply a choice of convention as before, restriction b) has been discussed previously, and restriction c) is similar.

We illustrate how one draws a representative of $[c] \in \mathcal{S}'(F_2)$ from its Dehn-Thurston parameter values in the following example.

Example 1: Consider the pants decomposition on F_2 as indicated in Figure 5*a*. The parameter value $(3, 1, 2) \times (2, -1, 0)$ represents the multiple curve in Figure 6. In an annular neighborhood of K_1, the curve corresponding to $(3, 1, 2) \times (2, -1, 0)$ twists twice to the right with three components, in an annular neighborhood of K_2 twists once to the left with one component, and in a neighborhood of K_3 there is no twisting with two components. One draws models in each annulus and connects them up uniquely using ε-translates of the arcs l_{ij}.

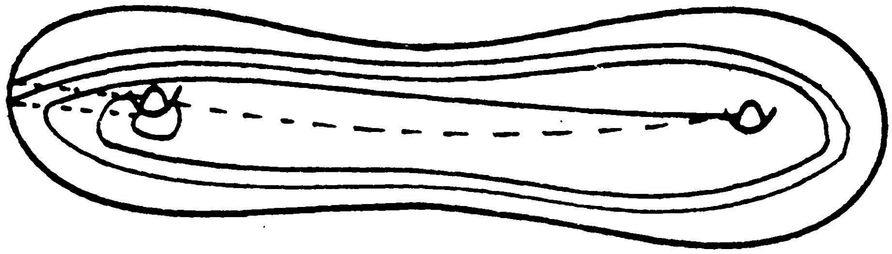

FIGURE 6

In this example, we give the parameter values for a connected multiple arc. There is no known algorithm for deciding if a given parameter value corresponds to a connected multiple arc. This is a hard combinatorial problem in general.

We are now in a position to give a precise statement to our main problem.

PROBLEM: *Compute the natural action of Lickorish's generators for $MC(F_g)$ on the Dehn-Thurston parameter values for $\mathcal{S}'(F_g)$.*

Before we describe how to attack this problem, let me indicate the nature of the results obtained. Regard our parameter space $(\mathbf{Z}^+)^{3g-3} \times \mathbf{Z}^{3g-3}$ as a subset of $\mathbf{R}^{6g-6}$ in the natural way. Given $[\phi] \in MC(F_g)$, there corresponds a *finite* decomposition $K_{[\phi]}$ of $\mathbf{R}^{6g-6}$ by cones based at the origin, and $[\phi]$ acts like an invertible integral matrix on the parameter values in each cone of $K_{[\phi]}$. Following Thurston [T1], we will call such an action on $\mathcal{S}'(F_g)$ *piecewise-integral-linear* (or *PIL*) transformation.

THEOREM. *The action of $MC(F_g)$ on $\mathcal{S}'(F_g)$ admits a faithful representation as a group of PIL transformations.*

In fact, the representation is faithful in the strong sense that there are $2g + 1$ multiple arcs so that $[\phi] = 1 \in MC(F_g)$ if and only if $[\phi]$ fixes each multiple arc in the collection. This immediately gives the following corollary.

COROLLARY. *There is a practical algorithm for solving the word problem in Lickorish's generators for* $MC(F_g)$.

Given a word w in Lickorish's generators, one simply considers the action of w on our collection of $2g+1$ multiple arcs, and $[w] = 1 \in MC(F_g)$ if and only if w fixes each multiple arc in our collection.

We prove the theorem and corollary by actually writing down formulas that describe the action of Lickorish's generators as in the problem, noting the *PIL* character and checking faithfulness. For convenience, we now restrict attention to the case of genus two. In this case, we choose the pants decomposition in Figure 5*a*. Three of the Lickorish generating curves (see Figure 2) are curves in this pants decomposition in this case, and we first investigate the action of Dehn twists along these on the Dehn-Thurston parametrization for multiple arcs.

Example 2: We compute the action on the curve $(3, 1, 2) \times (2, -1, 0)$ in example 1 of the Dehn twist $\tau_{K_3}^{+1}$ along the pants curve K_3 in Figure 5*a*. The result of this Dehn twist is shown in Figure 7. This curve has coordinates $(3, 1, 2) \times (2, -1, 2)$ and differs from $(3, 1, 2) \times (2, -1, 0)$ only in an annular neighborhood of K_3.

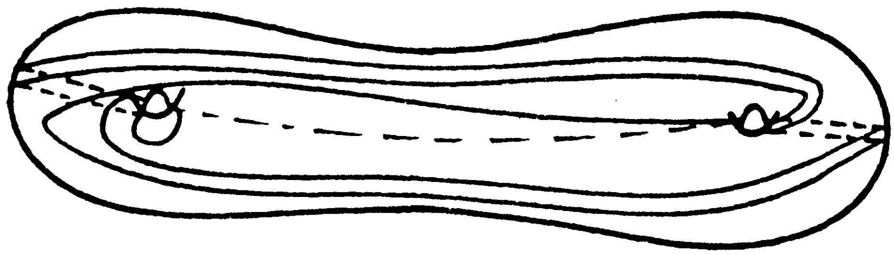

FIGURE 7

This example is typical, and a Dehn twist on the pants curve K_i acts on the Dehn-Thurston parametrization as the *linear* map

$$\tau_{K_i}^{\pm 1} : (m_1, \ldots, m_{3g-3}) \times (t_1, \ldots, t_{3g-3}) \\ \to (m_1, \ldots, m_{3g-3}) \times (t_1, \ldots, t_i \pm m_i, \ldots t_{3g-3}) .$$

This fact was noted by Dehn.

However, the action of Dehn twists along the other two curves in the Lickorish generating set are not nearly so simple. To tackle the problem of computing them, we note that these curves are curves in the pants decomposition indicated in Figure 5*c*. If we had a way of computing the Dehn-Thurston parameter values relative to the pants decomposition in Figure 5*c* from the parameter values relative to the pants decomposition in Figure 5*a* and vice-versa, then we would be able to compute the action of each of the Lickorish generators relative to the original pants decomposition in Figure 5*a*. This is in fact what we do. The philosophy comes from linear algebra: if a transformation (a Dehn twist) is hard to compute, change basis (pants decomposition).

We pass from Figure 5*a* to Figure 5*c* by means of two *elementary transformations*, which we now describe. The first one takes us from the pants decomposition in Figure 5*a* to the one in Figure 5*b*. It may also be described as the transformation pictured in Figure 8*b*; cutting along the right-most and left-most curves in Figures 5*a* and 5*b* gives us the surface pictured in Figure 8*b*. The second transformation takes us from the pants decomposition in Figure 5*b* to the one in Figure 5*c*. It may also be described by two applications of the transformation pictured in Figure 8*a*; cutting along the nullhomologous curves in Figures 5*b* and 5*c* gives us *two* copies of the surface pictured in Figure 8*a*.

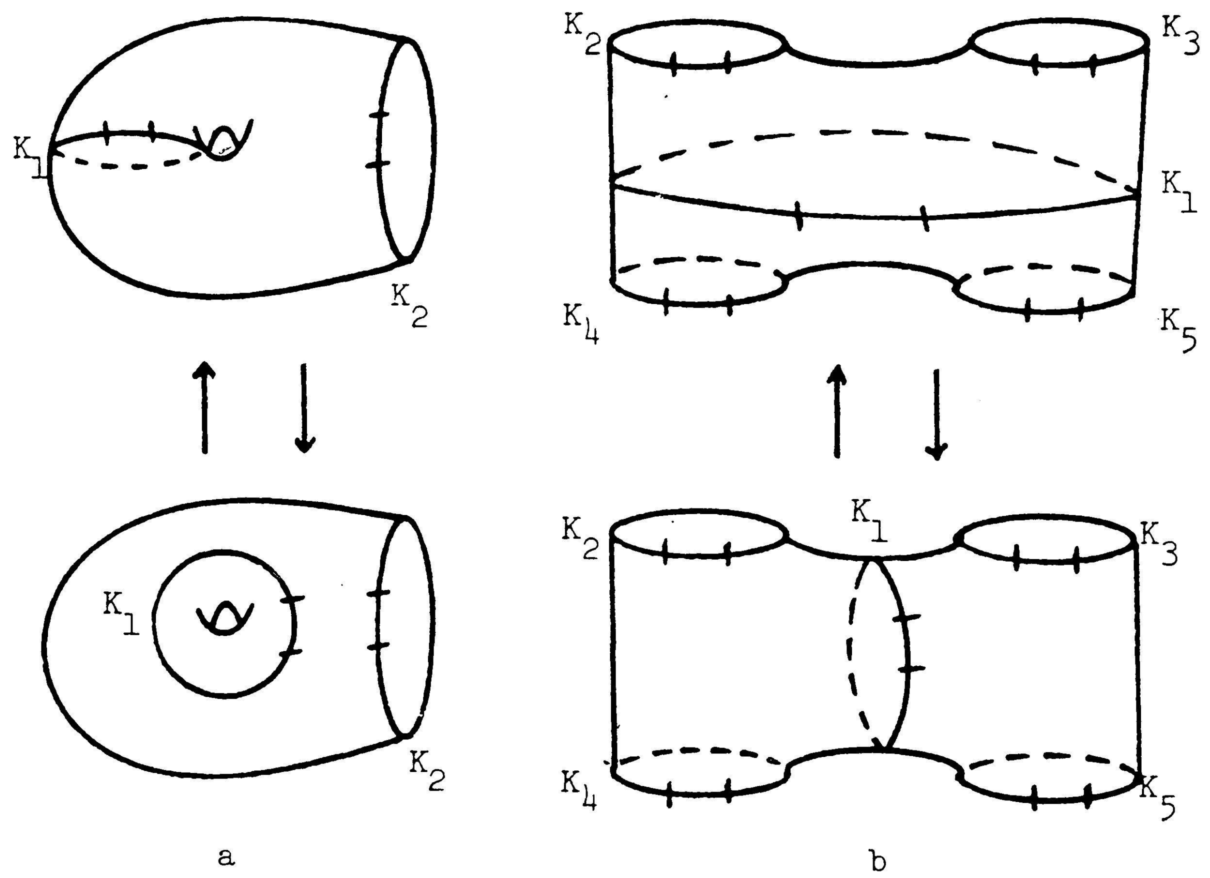

FIGURE 8

We will call the transformations pictured in Figure 8*a* and 8*b* the *first* and *second elementary transformations*, respectively. Thus, the computation of the action of $MC(F_2)$ on the collection of multiple curves in F_2 is reduced to the computation of the two elementary transformations. In fact, the same procedure works for surfaces of arbitrary genus; there exists a (finite) collection of pants decomposition of F_g, all related by sequences of elementary transformations, so that each of the Lickorish generating curves is a pants curve in at least one of the pants decompositions. Our problem reduces in general to the computation of the two elementary transformations. In fact, somewhat more is true. [HT] show that *any* two pants decompositions on F_g are related by sequences of our elementary transformations, but we will not need this stronger result.

The first elementary transformation is relatively easy and can be done by actually isotoping representatives of multiple arcs about on the torus-minus-a-disc. The second elementary transformation requires more work. The techniques we develop for the second elementary transformation also apply to the first elementary transformation, and we concentrate on the second elementary transformation for now.

The basic idea of the computation is to lift to an appropriate covering space. This on the one hand simplifies visualizing curves on surfaces and on the other introduces some complications. Let S_2 denote the sphere-minus-four-discs. We will define a regular planar cover $\pi_2 : \tilde{S}_2 \to S_2$ below. If $[c] \in \mathscr{S}'(S_2)$ is a multiple arc, we orient the components of c arbitrarily and choose some lift $\tilde{c}$ of c to $\tilde{S}_2$. We will isotope $\tilde{c}$ about in $\tilde{S}_2$ to some $\tilde{\tilde{c}}$ and define $\bar{c}$ to be the projection of $\tilde{\tilde{c}}$ by π_2. The isotopy in S_2 is chosen in such a way that c looks locally like the appropriate Dehn-Thurston model. We cannot guarantee that this isotopy is π_2-equivariant, so $\bar{c}$ is not necessarily embedded. Note, however, that $\bar{c}$ is at least homotopic to the embedding c. All our computations will take place in the total space $\tilde{S}_2$ because we gain a facility in picturing isotopies of curves. However, to get around the problem that $\bar{c}$ is not in general embedded requires a great deal of hard combinatorial work which we will suppress in this exposition. One is required to consider one-manifolds *immersed* in surfaces.

The cover $\pi_2 : \tilde{S}_2 \to S_2$ can be described as follows. Let Λ_2 be the group generated by rotations-by-π about the integral points $\mathbf{Z}^2$ in $\mathbf{R}^2$. The action of Λ_2 on $\mathbf{R}^2 \backslash \mathbf{Z}^2$ describes a cover of the four-times punctured sphere by $\mathbf{R}^2 \backslash \mathbf{Z}^2$. Let N be a small, π_2-invariant, diamond-shaped open neighborhood of $\mathbf{Z}^2$ in $\mathbf{R}^2$, as indicated in Figure 9*a*. The action of Λ_2 on $\mathbf{R}^2 \backslash N$

gives a cover of S_2 by $\mathbf{R}^2 \backslash N$, denoted $\tilde{S}_2$. Cutting S_2 along the arcs $a_1, \ldots, a_4$ in S_2 indicated in Figure 9*b* decomposes S_2 into two octagons, labeled f and b in Figure 9*b*. The lifts to $\tilde{S}_2$ of these octagons give a tiling of $\tilde{S}_2$; if we are careful in the choice of the arcs a_i, then we can guarantee that the associated tiling is regular. This regular tiling of $\tilde{S}_2$ by octagons is indicated in Figure 9*c*; the corresponding tiling of the plane by squares and octagons is a popular architectural motif and can be seen, for instance, in the Park Street Subway Station in Boston. We number the boundary components of S_2 as indicated in Figure 9*b*; we put a number inside each component C of ∂N to indicate the boundary component of S_2 twice covered by ∂C, as indicated in Figure 9*c*.

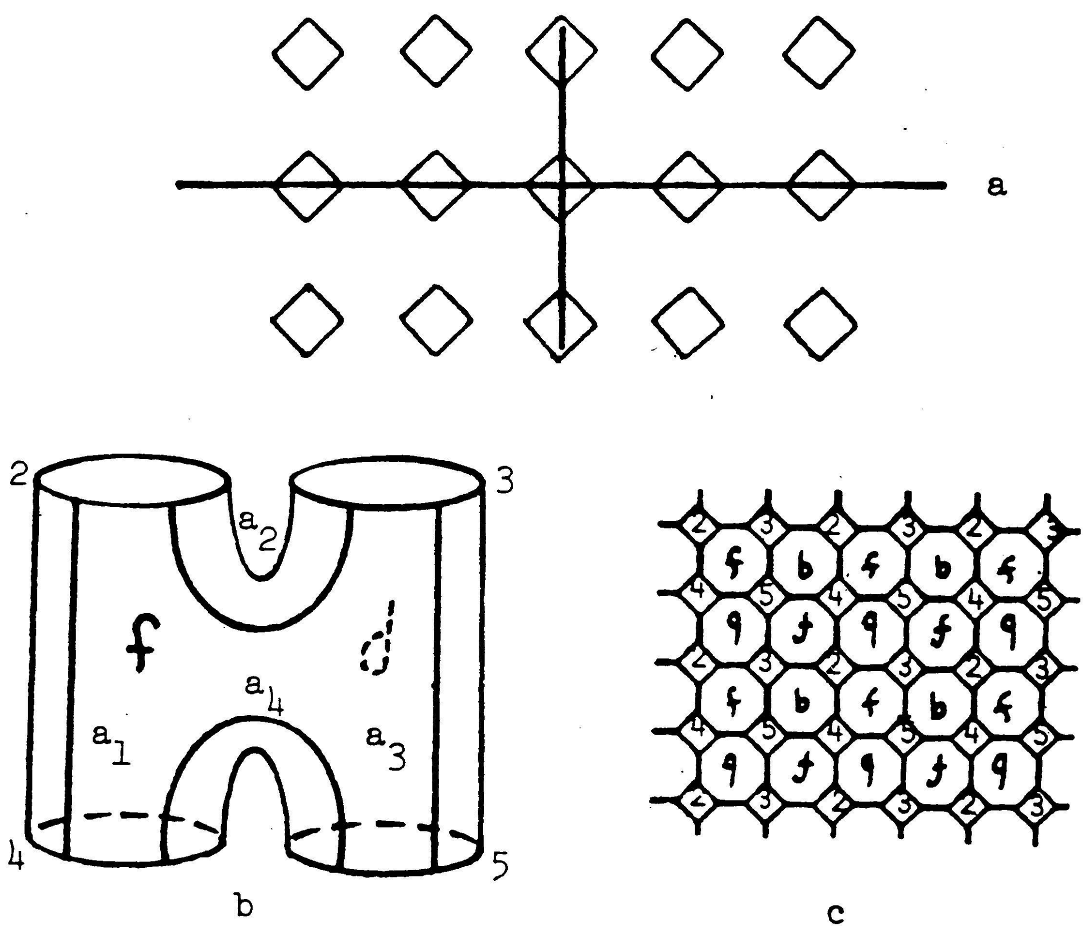

FIGURE 9

Example 3: In this example, we compute the action on the curve (3, 1, 2) × (2, −1, 0) in example 1 of a right twist along the curve K_2 in Figure 5*b*. At each stage of the computation, we illustrate the curve on F_2, the associated multiple arc on S_2 and the lift to $\tilde{S}_2$. In Figure 10*a*, we

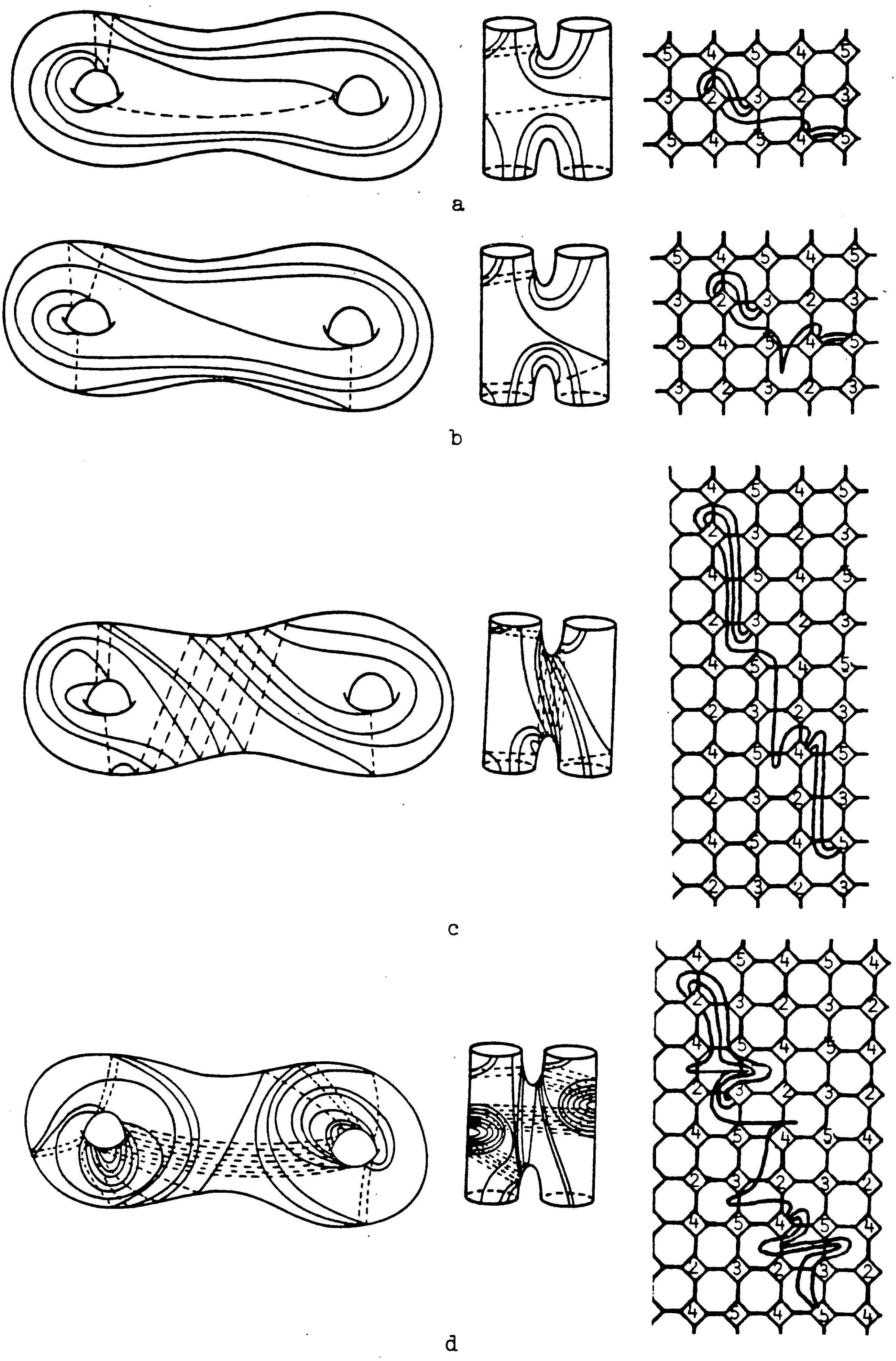

FIGURE 10

illustrate the curve $(3, 1, 2) \times (2, -1, 0)$, and in Figure 10*b*, we illustrate the isotopic curve with coordinates $(3, 6, 2) \times (1, 0, 0)$ for the choice of pants decomposition in Figure 5*b*. In Figure 10*c*, we illustrate the curve $(3, 6, 2) \times (1, 6, 0)$, which is the image of $(3, 6, 2) \times (1, 0, 0)$ under the Dehn twist along the pants curve K_2 in Figure 5*b*. In Figure 10*d*, we illustrate the curve with coordinates $(3, 13, 2) \times (4, -7, 2)$ relative to the first pants decomposition and isotopic to $(3, 6, 2) \times (1, 6, 0)$. This example indicates some typical phenomena of the second elementary transformation.

We have introduced this example for several reasons. It indicates the general technique of lifting our computation to the total space $\tilde{S}_2$ and shows the facility gained in visualizing the second elementary transformation as an isotopy in $\tilde{S}_2$. The example also shows that our computation has content, for even this relatively simple case of the action of a single Dehn twist on a connected multiple arc is reasonably complicated. Finally, in any specific example such as this, it is easy to guarantee that our curves are embedded in S_2 at each stage of the computation; however, to prove that our techniques work in general requires a lot more work.

We will introduce a bit of notation and then give the formulas for the elementary transformations; in fact, we will introduce a new parametrization for multiple arcs. Given a pants decomposition as before, we will keep track of the Dehn-Thurston twisting parameters exactly as before. However, instead of keeping track of the Dehn-Thurston intersection numbers, we will record the number of arcs in each embedded pair of pants that are ε-translates of the various arcs l_{ij} in Figure 5. It is obvious how one can pass back and forth between the Dehn-Thurston parameter values and our new parameter values.

In the surface S_2 with the horizontal pants decomposition shown in Figure 8*b*, top, we will denote the number of arcs parallel to the various l_{ij}. $1 \leqslant i \leqslant j \leqslant 3$, by l_{ij} in the top pair of pants and by k_{ij} in the bottom pair of pants. Similarly, for the vertical pants decomposition shown in Figure 8*b*, bottom, we will denote the number of arcs parallel to the various l_{ij} by l'_{ij} in the left-most pair of pants, and by k'_{ij} in the right-most pair of pants. The twisting numbers will be denoted by t_i for the horizontal pants decomposition, and by t'_i for the vertical pants decomposition, with the pants curves numbered as in Figure 8. Let $\wedge$ denote the infimum, $\vee$ the supremum, and let $K = \mathrm{k}_{11} + t_1$ and $L = \mathrm{l}_{11} + t_1$.

THEOREM. *The second elementary transformation is given by the following formulas.*

$$
\begin{aligned}
k'_{11} &= k_{22} + l_{33} + (L-k_{13}) \vee 0 + (-L-l_{12}) \vee 0 \\
k'_{22} &= \big(L \wedge l_{11} \wedge (k_{13}-l_{12}-L)\big) \vee 0 \\
k'_{33} &= \big(-L \wedge k_{11} \wedge (l_{12}-k_{13}+L)\big) \vee 0 \\
k'_{23} &= \big(k_{13} \wedge l_{12} \wedge (k_{13}-L) \wedge (l_{12}+L)\big) \vee 0 \\
k'_{12} &= -2k'_{22} - k'_{23} + k_{13} + k_{23} + 2k_{33} \\
k'_{13} &= -2k'_{33} - k'_{23} + l_{12} + l_{23} + 2l_{22} \\
l'_{11} &= k_{33} + l_{22} + (K-l_{13}) \vee 0 + (-K-k_{12}) \vee 0 \\
l'_{22} &= \big(K \wedge k_{11} \wedge (l_{13}-k_{12}-K)\big) \vee 0 \\
l'_{33} &= \big(-K \wedge l_{11} \wedge (k_{12}-l_{13}+K)\big) \vee 0 \\
l'_{23} &= \big(l_{13} \wedge k_{12} \wedge (l_{13}-K) \wedge (K+k_{12})\big) \vee 0 \\
l'_{12} &= -2l'_{22} - l'_{23} + l_{13} + l_{23} + 2l_{33} \\
l'_{13} &= -2l'_{33} - l'_{23} + k_{12} + k_{23} + 2k_{22} \\
t'_2 &= l_{33} + \big((l_{13}-l'_{23}-2l'_{22}) \wedge (K+l'_{33}-l'_{22})\big) \vee 0 + t_2 \\
t'_3 &= -k'_{33} + \big((L+k'_{33}-k'_{22}) \vee -(l_{12}-k'_{23}-2k'_{33})\big) \wedge 0 + t_3 \\
t'_4 &= -l'_{33} + \big((K+l'_{33}-l'_{22}) \vee -(k_{12}-l'_{23}-2l'_{33})\big) \wedge 0 + t_4 \\
t'_5 &= k_{33} + \big((k_{13}-k'_{23}-2k'_{22}) \wedge (L+k'_{33}-k'_{22})\big) \vee 0 + t_5 \\
t'_1 &= k_{22} + l_{22} + k_{33} + l_{33} - \big(l'_{11}+k'_{11}+(t'_2-t_2)+(t'_5-t_5)\big) \\
&\quad + \operatorname{sgn}(L+K+l'_{33}-l'_{22}+k'_{33}-k'_{22})\,(t_1+l'_{33}+k'_{33})
\end{aligned}
$$

sgn(0) is defined by the following formula.

$$
\operatorname{sgn}(0) = \begin{cases} +1, & l_{12} - 2k'_{33} - k'_{23} \neq 0 \\ -1, & \text{else} \end{cases}
$$

The inverse transformation, from primed to unprimed variables, can be computed by using a symmetry of the surface S_2.

For completeness, we include the formulas that describe the first elementary transformation. The unprimed variables l_{ij}, t_1, t_2 describe the parameter values relative to the meridinal pants decomposition of the torus-minus-a-disc, (Figure 8*a*, top) and the primed variables $\mathrm{l}_{ij}, t'_1, t'_2$ describe the parameter values relative to the longitudinal pants decomposition (Figure 8*a*, bottom). r denotes the value of $\mathrm{l}_{12} = \mathrm{l}_{13}$.

THEOREM. *The first elementary transformation is given by the following formulas.*

$$\begin{aligned}
\mathrm{l}'_{11} &= (r-|t_1|) \vee 0 \\
r' &= \mathrm{l}'_{12} = \mathrm{l}'_{13} = (r-\mathrm{l}'_{11}) + \mathrm{l}_{11} \\
\mathrm{l}'_{23} &= \big(|t_1|-(r-\mathrm{l}'_{11})\big) \\
t'_2 &= t_2 + \mathrm{l}_{11} + \big((r-\mathrm{l}'_{11}) \wedge t_1\big) \vee 0 \\
t'_1 &= -\operatorname{sgn}(t_1)\,\big(\mathrm{l}_{23}+(r-\mathrm{l}'_{11})\big)
\end{aligned}$$

In these formulas, $\operatorname{sgn}(0) = -1$. The inverse transformation can easily be solved for algebraically.

These theorems give explicit formulas for the action of Lickorish's generators of $MC(F_g)$ on $\mathscr{S}'(F_g)$ as described previously. The *PIL* character of the action is directly implied by these theorems. Unfortunately, the formulas are rather cumbersome, insofar as several of the Lickorish generators act as linear maps conjugated by compositions of the elementary transformations.

One's first reaction to the complexity of the situation is panic, and an appropriate response is to write a computer code to perform the algebra of the computations. The formulas of the elementary transformations are particularly amenable to computerization, since they are essentially sums of infs and sups of linear maps. The notable exception to this is the sign that appears in the expression for the twisting number t_1 in either transformation.

A FORTRAN code has been written to compute the action of $MC(F_g)$ on the collection of multiple arcs as described in this paper. Roughly a million cases of the computation have been run, checking that a transformation followed by its inverse yields the identity in each case (as one would hope!). In a computation of this magnitude, there is a potential for algebraic mistakes, and the code was originally written to check that all the components of the computation were working properly; at this point, I am quite confident that the formulas above are error-free. Moreover, many trends predicted by Thurston's theory of surface automorphisms

[T_1, T_2, FLP] are exhibited by experimenting with this code, so it is instructive to play with.

I should remark that though we have restricted attention to the action of $MC(F_g)$ on closed multiple arcs in F_g, the computations in this paper apply more generally to any surface of negative Euler characteristic. The surface may be bounded, non-compact, or even non-orientable, provided we require that multiple arcs be two-sided.

As a final remark, let me mention that Thurston introduced a space $\mathscr{PF}(F)$ of "projective measured foliations" on F which is central to his treatment of surface automorphisms. (See [FLP].) $\mathscr{PF}(F)$ forms a boundary for a compactification of the Teichmuller space $\mathscr{T}(F)$ of F, and the (discrete) set $\mathscr{S}'(F)$ which we treat here embeds in a natural way as a dense subset of the (connected) space $\mathscr{PF}(F)$. The compactification of $\mathscr{T}(F)$ by $\mathscr{PF}(F)$ is natural in the sense that the usual action of $MC(F)$ on $\mathscr{T}(F)$ extends continuously to the natural action of $MC(F)$ on $\mathscr{PF}(F)$. (See [K].) Thurston generalized Dehn's parametrization of $\mathscr{S}'(F)$ to a parametrization of $\mathscr{PF}(F)$, and the formulas given in this paper apply to this setting to give the action of $MC(F)$ on Thurston's parametrization of $\mathscr{PF}(F)$. Thus, the formulas derived here in fact describe the action of $MC(F)$ on Thurston's boundary for $\mathscr{T}(F)$. There are other formulations of the theory using "measured geodesic laminations" and "measured train tracks" (See [HP].) together with appropriate parametrizations, and our computations also apply to these settings.

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