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A LINEAR ALGEBRA PROOF OF CLIFFORD'S THEOREM

by W. J. Gordon

One of the central results in the theory of algebraic curves is the Riemann-Roch theorem. This theorem guarantees that for divisors D of large degree on a curve the dimension of the associated linear system $| D |$ and the degree of the divisor differ by a constant, the genus of the curve. Clifford's theorem complements Riemann-Roch, by giving information about $\dim | D |$ when the degree of D is small.

The standard modern proof of the Riemann-Roch theorem is a cohomological, scheme-theoretic one. However, elementary proofs are often given because of the importance of the result in the classical theory of algebraic curves. In contrast, Clifford's theorem, which complements Riemann-Roch and provides useful information about hyperelliptic curves, is usually given only a scheme-theoretic proof, and so is not widely known.

In this paper, I give an elementary proof of Clifford's theorem. First I prove a key result, Clifford's lemma, which has the flavor of linear algebra although it is actually a result in algebraic geometry. Clifford's lemma and the Riemann-Roch theorem provide an easy proof of the first part of Clifford's theorem; the other two parts follow by linear algebra arguments.

The proof of the third part of the theorem depends only on facts about divisors on hyperelliptic curves. This proof emphasizes the view of a hyperelliptic curve as a double covering of the projective line. In contrast, the usual proof relies on the characterization of hyperelliptic curves in terms of the canonical morphism $C \to \mathbf{P}_{g-1}$.

1. The key lemma

For this section, let K be a field and let A, B, and C be vector spaces over K. Let r, and s, denote the dimensions of the vector spaces A and B.

Definition. The K-bilinear map $\varphi : A \times B \to C$ is *bi-injective* if the induced maps $\varphi(a, \ ) : B \to C$ and $\varphi(\ , b) : A \to C$ are injective whenever

$a \neq 0$ and $b \neq 0$. Equivalently, φ is bi-injective if $\varphi(a, b) = 0$ implies a or b is zero.

The image of the bi-injective map $\varphi: A \times B \to C$ is not in general a vector subspace of C, but this image contains an s-dimensional family, $\mathscr{F} = \{A_b = \varphi(A, b) \mid b \in B\}$, of r-dimensional vector subspaces of C. Since C contains this family $\mathscr{F}$, one would expect that $\dim C$ is at least $r + s$. This is not the case at all! One has

CLIFFORD'S LEMMA. *Let K be algebraically closed, and let $\varphi: A \times B \to C$ be bi-injective. Then*

$$\dim C \geqslant r + s - 1 .$$

Example 1. The lower bound given can occur. For example, let $P_n = \{\text{polynomials in } K[x] \text{ of degree} \leqslant n\}$. Then multiplication of polynomials defines a bi-injective map $\mu: P_r \times P_s \to P_{r+s}$, for which equality holds in Clifford's Lemma.

Example 2 (Schanuel). If K is not algebraically closed the result is false. Namely, let F be an extension field of E, of degree $n > 1$. Then the multiplication map $\mu: F \times F \to F$ is a bi-injective map of E-vector spaces, yet Clifford's Lemma would imply

$$\dim F = n > 2 \dim F - 1 = 2n - 1 .$$

(This example shows that the Lemma is not actually a result of linear algebra.)

Proof of Clifford's Lemma. Assume that $\dim C = t \leqslant r + s - 2$. Let $\{a_1, \ldots, a_r\}$ be a basis for A, $\{b_1, \ldots, b_s\}$ one for B, and $\{c_1, \ldots, c_t\}$ one for C. I will show that there are elements $a \in A$, $b \in B$ both nonzero for which $\varphi(a, b) = 0$. Writing $a = \sum_{ij} \alpha_i a_i$ and $b = \sum \beta_j b_j$, bilinearity shows that

$$\varphi(a, b) = \sum_{ij} \alpha_i \beta_j \varphi_{ij} \quad \text{where} \quad \varphi_{ij} = \varphi(a_i, b_j) = \sum_k \lambda_k^{ij} c_k .$$

Then, $\varphi(a, b) = 0$ if and only if

$$(*) \qquad \sum_{ij} \alpha_i \beta_j \lambda_k^{ij} = 0 \quad \text{for} \quad k = 1, \ldots, t .$$

Since a and b are nonzero, their coordinate tuples $(\alpha_1, \ldots, \alpha_r)$ and $(\beta_1, \ldots, \beta_s)$ can be viewed as points in the projective spaces $\mathbf{P}_{r-1}$ and $\mathbf{P}_{s-1}$. The Segré embedding $\sigma: \mathbf{P}_{r-1} \times \mathbf{P}_{s-1} \to \mathbf{P}_{rs-1}$ given by

$$\sigma((\alpha_1, ..., \alpha_r), (\beta_1, ..., \beta_s)) = (\alpha_1\beta_1, ..., \alpha_1\beta_s, \alpha_2\beta_1, ..., \alpha_2\beta_s, ..., \alpha_r\beta_s)$$

is a projective morphism establishing an isomorphism between $\mathbf{P}_{r-1} \times \mathbf{P}_{s-1}$ and the image $\mathscr{S} = \sigma(\mathbf{P}_{r-1} \times \mathbf{P}_{s-1})$. [4, Ex I.2.14] Once I label the coordinates of $\mathbf{P}_{rs-1}$ as $(z_{11}, ..., z_{1s}, z_{21}, ..., z_{2s}, ..., z_{rs})$, $\mathscr{S}$ can be identified with the algebraic subset of $\mathbf{P}_{rs-1}$ cut out by the polynomials

$$\{z_{ij}\, z_{pq} - z_{iq}z_{pj} \mid 1 \leqslant i, p \leqslant r \text{ and } 1 \leqslant j, q \leqslant s\}\,.$$

$\mathscr{S}$ is an algebraic subvariety of $\mathbf{P}_{rs-1}$, of dimension $r + s - 2$.

In $\mathbf{P}_{rs-1}$ we can also consider the algebraic subvariety $\mathscr{T}$ cut out by the polynomials $\{\sum_{ij} z_{ij}\lambda_k^{ij} \mid 1 \leqslant k \leqslant t\}$. Since $\mathscr{T}$ is cut out by $t \leqslant r + s - 2$ equations and $\dim \mathscr{S} = r + s - 2$, $\mathscr{S}$ and $\mathscr{T}$ have a nonempty intersection, all of whose components have dimension at least $(r+s-2) - t$, which is $\geqslant 0$. [4, p. 48] However, any intersection point of $\mathscr{S}$ and $\mathscr{T}$ corresponds to a pair of points $(\alpha_1, ..., \alpha_r) \in \mathbf{P}_{r-1}, (\beta_1, ..., \beta_s) \in \mathbf{P}_{s-1}$ satisfying (*). The corresponding points $a = \Sigma\, \alpha_i a_i \in A$, $b = \Sigma\, \beta_j b_j \in B$ are nonzero, yet $\varphi(a, b) = 0$. Since this contradicts the bi-injectivity of φ, I have shown that

$$\dim C \geqslant r + s - 1\,. \qquad \square$$

The assumption that K is algebraically closed was only needed to guarantee that $\mathscr{S} \bigcap \mathscr{T}$, which by dimension theory corresponds locally to a proper ideal, was nonempty. Hilbert's Nullstellensaltz shows that any proper ideal in a polynomial ring over an algebraically closed field cuts out at least one point.

2. A BRIEF RESUME OF DIVISORS ON CURVES

In this section, I will establish notation for divisors, and state the Riemann-Roch theorem. Let C be a nonsingular projective algebraic curve defined over an algebraically closed field K. C is contained in some projective space $\mathbf{P}_N$ over K, and a (closed) *point* of C is any closed point $(p_0, ..., p_N)$ of $\mathbf{P}_N$ at which all the polynomials cutting out C vanish. The *group of divisors on C* is the free abelian group generated by the points of C. Any divisor can be written in the form

$$N = \Sigma\, n_P \cdot P$$

where the n_P are integers, almost all zero. The *degree* of N is the integer $\deg N = \Sigma\, n_P$. The divisor N is *effective* if all the n_P are $\geqslant 0$; this is written as $N \succ 0$. I write $D \succ E$ to mean $D - E \succ 0$.

To any function f on C one can associate a divisor $(f) = \Sigma \operatorname{ord}_P(f) \cdot P$, where $\operatorname{ord}_P(f)$ is the order of zero or pole of f at P. For any function f, the divisor (f) has degree 0. The divisors D, E are *linearly equivalent*, denoted by $D \sim E$, if for some function f, $D - E = (f)$. To a divisor D on C one can associate a set of functions on C,

$$L(D) = \{\text{functions } f \text{ on } C \mid (f) + D \succ 0\} \bigcup \{0\}.$$

Then $L(D)$ is a K-vector space of dimension $l(D)$; the set $|D| = \{$divisors $E \sim D \mid E \succ 0\}$ of the divisors $(f) + D$ corresponding to functions f in $L(D)$ is the *linear system* associated to D. If $\{f_0, ..., f_n\}$ is a basis of $L(D)$, then $|D|$ can be identified with $\mathbf{P}_n$ by associating the divisor

$$(a_0 f_0 + ... + a_n f_n) + D$$

to the triple $(a_0, ..., a_n)$; one writes $\dim |D|$ for the dimension of this projective space. To define $\dim |D|$ intrinsically, notice that $\dim |D| \geqslant r$ if and only if, for all points $P_1, ..., P_r$ in C, there is a divisor E in $|D|$ of the form $E = P_1 + ... + P_r + Q$, with Q effective. Any such divisor E is necessarily effective and linearly equivalent to D, and has support containing each P_i. (In fact, since $\dim |D| \geqslant r$ there is a linearly independent set $\{f_0, ..., f_r\}$ of functions in $L(D)$. One can choose E of the form $E = D + (\alpha_0 f_0 + ... + \alpha_r f_r)$ for some $\alpha_0, ..., \alpha_r \in K$.)

If $D \sim E$, then $|D| = |E|$, so $\dim |D| = \dim |E|$, and $L(D)$ is isomorphic to $L(E)$. Since for any function f on C $\deg (f) = 0$, also $\deg D = \deg E$. In particular, if $\deg D < 0$ then $|D|$ is empty, and $L(D) = (0)$.

Definition. The curve C *admits a* g_n^r if there exists a divisor D on C of degree n, and with $\dim |D| = r$. We call $|D|$ *the* g_n^r *defined by* D.

Notice that if D defines a g_n^r and $E \sim D$, then E defines the same g_n^r. Yet a curve may admit several distinct g_n^r's if it contains non-linearly equivalent divisors all defining g_n^r's. To explain the notation, assume that $L(D)$ has basis $(f_0, ..., f_r)$. Then the map

$$P \to (f_0(P), ..., f_r(P))$$

is a rational map from C into $\mathbf{P}_r$, defined except at the common zeros of all the f_i (the "fixed points" of $|D|$); via this map, the pullback of every hyperplane in $\mathbf{P}_r$ is a divisor on C of degree n. [4, II: 7.7 and 7.8.1]

The Riemann-Roch Theorem defines for each curve two invariants—a nonnegative integer g, the *genus*, and a divisor $\mathscr{K}$, the *canonical divisor* (determined only up to linear equivalence). [For a modern proof, cf. 4, Ch. IV.1; an elementary proof is given in 2].

THEOREM (Riemann-Roch). *Let* C *be a projective nonsingular algebraic curve. The genus of* C *is a nonnegative integer* g. *For all divisors* D *on* C,

$$\dim | D | \geqslant \deg D - g .$$

If the strict inequality holds, D *is* special. *For all special divisors* D,

$$\dim | D | = \deg D + 1 - g + \dim | \mathcal{K} - D | .$$

COROLLARY. $\deg \mathcal{K} = 2g - 2$; $\dim | \mathcal{K} | = g - 1$; *and all divisors* D *of degree* $> 2g - 2$ *are nonspecial.*

3. CLIFFORD'S THEOREM — THE ELEMENTARY PROOF

Clifford's Theorem complements Riemann-Roch by providing information about special divisors, which of necessity are of small degree. The theorem also gives a sufficient condition that the curve C is hyperelliptic. (The theorem owes its name to the appearance of its first part in [1].) The proof I give here is elementary; more typical modern proofs [e.g. 4, Ch. IV, section 5 and 3, Ch. 2, section 3] involve considering whether the canonical morphism $C \to \mathbf{P}_{g-1}$ defined by the canonical divisor $\mathcal{K}$ is an embedding.

Definition. C is a *hyperelliptic* curve if its genus g is at least 2, and if C admits a g^1_2.

Remarks.

1. C is hyperelliptic if and only if there is a rational map $C \to \mathbf{P}_1$ of degree 2.

2. This happens if and only if C has an (affine) equation of the form $y^2 = f(x)$.

3. Part (3) of Clifford's Theorem shows that a hyperelliptic curve has a unique g^1_2. Contrast this to the case of an elliptic curve, where $g = 1$. Here any divisor of degree 2 defines a g^1_2. Yet choosing distinct points P, Q one sees easily that the divisors $2P$ and $P + Q$ are not linearly equivalent, and so define distinct g^1_2's.

THEOREM (Clifford). *Let* C *be a curve of genus* g, *and let* D *be an effective special divisor on* C. *Then*

(1) $\dim|D| \leqslant \frac{1}{2}\deg D$.

(2) *Equality holds in only 3 cases:* (a) $D = 0$; *or*

(b) $D = \mathscr{K}$; *or*

(c) *C is a hyperelliptic curve.*

(3) *If Case 2c holds then C admits a unique g^1_2, $\deg D = 2r$ for some integer $r \geqslant 1$, and $D \sim r \cdot g^1_2$.*

Proof of (1). Since D is effective special, the vector spaces $L(D)$ and $L(\mathscr{K}-D)$ are both of positive dimension. Define a map $\mu: L(D) \times L(\mathscr{K}-D) \to L(\mathscr{K})$ by $\mu(f, g) = f \cdot g$. (Since $(f) + D > 0$ and $(g) + \mathscr{K} - D > 0$, $(fg) + \mathscr{K} = (f) + (g) + \mathscr{K} = [(f)+D] + [(g)+\mathscr{K}-D] > 0$ so $fg \in L(\mathscr{K})$.) This map is bi-injective, so $\dim L(\mathscr{K}) \geqslant \dim L(D) + \dim L(\mathscr{K}-D) - 1$ by Clifford's Lemma. Since $l(D) = \dim|D| - 1$, one has

$$(1) \qquad \dim|\mathscr{K}| \geqslant \dim|D| + \dim|\mathscr{K} - D|.$$

On the other hand, Riemann-Roch guarantees that

$$(2) \qquad \deg D + 1 - g = \dim|D| - \dim|\mathscr{K} - D|.$$

Adding these, and recalling that $\dim|\mathscr{K}| = g - 1$, one gets $\deg D \geqslant 2\dim|D|$. □

Implicit in the proof is a result I will need later.

LEMMA 1. *For the effective special divisor D, $\dim|D| = \frac{1}{2}\deg D$ if and only if $\dim|\mathscr{K}| = \dim|D| + \dim|\mathscr{K} - D|$. This holds if and only if $g - 1 \leqslant \dim|D| + \dim|\mathscr{K} - D|$. Further, equality holds for D if and only if it holds for (any effective divisor linearly equivalent to) $\mathscr{K} - D$.* □

Proof of (2). Assume that equality holds, and that D is neither 0 nor $\mathscr{K}$. Notice that if $\deg D = 2$, or $\deg \mathscr{K} - D = 2$, then D, or $\mathscr{K} - D$, defines a g^1_2 and C is hyperelliptic. Thus, I may assume that $\deg D$ and $\deg \mathscr{K} - D$ are both at least 4, so $\dim|D|$ and $\dim|\mathscr{K} - D|$ are both at least 2. Fix a point P in C. Since $\dim|\mathscr{K} - D| \geqslant 2$ I can choose a divisor $E = P + \Sigma e_R R$ in $|\mathscr{K} - D|$. Now fix a point Q on C but not in the support of E (i.e. $e_Q = 0$). Because $\dim|D| \geqslant 2$ I can choose a divisor (sloppily I call it D) in $|D|$ whose support contains both P and Q,

$$D = P + Q + \Sigma d_R R.$$

Set $I = \inf(D, E)$ and $S = \sup(D, E)$. Then

$$I = \Sigma \min(d_P, e_P) \cdot P \quad \text{and} \quad S = \Sigma \max(d_P, e_P) \cdot P .$$

Since P is in I, and Q is not, we have $0 < \deg I < \deg D$. Once I show that $\dim |I| = \frac{1}{2} \deg I$, by descent I will have shown that C is hyperelliptic.

Notice that $L(I) = L(D) \bigcap L(E)$. The inclusion $L(I) \subset L(D) \bigcap L(D)$ holds because $I < D$ and $I < E$. On the other hand, if $f \in L(D) \bigcap L(E)$, $(f) + D$ and $(f) + E$ are both effective. Then, for all points R, $\mathrm{ord}_R(f) \geqslant -d_R$ and $\mathrm{ord}_R(f) \geqslant -e_R$, so $\mathrm{ord}_R(f) + \min(d_R, e_R) \geqslant 0$ and $f \in L(I)$. Similarly, one sees that $L(D) + L(E) \subset L(S)$. Since $D < S$ and $E < S$ both $L(D)$ and $L(E)$ are subspaces of $L(S)$. If $\delta \in L(D)$ and $\varepsilon \in L(E)$, then for all R, $\mathrm{ord}_R(\delta+\varepsilon) \geqslant \min(\mathrm{ord}_R(\delta), \mathrm{ord}_R(\varepsilon)) \geqslant \min(-d_R, -e_R) = -\max(d_R, e_R)$. This shows that $\delta + \varepsilon \in L(S)$.

As subspaces of $L(S)$, we see that

$$\dim L(D) + \dim L(E) = \dim L(I) + \dim \big(L(D) + L(E)\big) .$$

Rewriting this in terms of linear systems gives

$$\dim |D| + \dim |E| \leqslant \dim |I| + \dim |S| .$$

Since $E \sim \mathscr{K} - D$, Lemma 1 applied to D gives

$$\dim |\mathscr{K}| \leqslant \dim |I| + \dim |S| .$$

Yet $I + S = D + E \sim \mathscr{K}$, so $S \sim \mathscr{K} - I$. Lemma 1, now applied to I, shows that $\dim |I| = \frac{1}{2} \deg I$. □

To prove the third part of the theorem I need some technical lemmas. We may assume that the curve C is hyperelliptic and so comes equipped with a given g^1_2. On any such curve I can define a function $\pi : C \to C$, by defining $\pi(P)$ to be the unique point Q such that $P + Q$ is a divisor in the given g^1_2. To verify that $\pi(P)$ is well defined, notice that if $P + Q$ and $P + R$ both belong to the given g^1_2, then $Q \sim R$. Since $g > 0$, Q must equal R [4, II. 6.10.1]; this shows that $\pi(P)$ is well-defined. Notice that since $\pi P + P$ is in the g^1_2, $\pi(\pi P) = P$.

Lemma 2. *For any point* P, $L(\mathscr{K} - P) = L(\mathscr{K} - P - \pi P)$ *and* $l(\mathscr{K} - P) = l(\mathscr{K}) - 1$.

Proof. $P + \pi(P)$ is a g^1_2 so $\dim |P + \pi P| = 1$ and by Lemma 1, $1 + \dim |\mathscr{K} - P - \pi P| = \dim |\mathscr{K}|$. Since $\mathscr{K} - P - \pi P < \mathscr{K} - P$

$< \mathscr{K}$, one sees that $L(\mathscr{K}-P-\pi P) \subset L(\mathscr{K}-P) \subset L(\mathscr{K})$. To prove $L(\mathscr{K}-P) = L(\mathscr{K}-P-\pi P)$ it suffices to show that $L(\mathscr{K}-P) \neq L(\mathscr{K})$. Yet if these were equal, the divisor P would be an effective special divisor of degree 1 with $\dim|\mathscr{K}-P| = \dim|\mathscr{K}|$. By Lemma 1, then $\dim|P|$ would equal $\frac{1}{2}\deg P$, which is absurd! □

Definition. The points $P_1, ..., P_k$ on C form a *disjoint set of points* if for each i, $P_i \neq \pi(P_i)$ and if the divisors $P_i + \pi P_i$ are pairwise disjoint.

LEMMA 3. *Let* $\{P_1, ..., P_n\}$ *be a disjoint set of points, with* $n \leqslant g$. *Then*

$$\dim \bigcap_1^n L(\mathscr{K}-P_i) = l(\mathscr{K}) - n = g - n\,.$$

Proof. Since $l(\mathscr{K}-P_i) = l(\mathscr{K}) - 1$, the intersection has dimension $\geqslant l(\mathscr{K}) - n$. Choose points $P_{n+1}, ..., P_g$ such that $\{P_1, ..., P_g\}$ is a disjoint set. Then

$$\bigcap_1^g L(\mathscr{K}-P_i) = \bigcap_1^g L(\mathscr{K}-P_i-\pi P_i) = L\Big(\mathscr{K}-\sum_1^g (P_i+\pi P_i)\Big)\,.$$

If $\dim \bigcap_1^n L(\mathscr{K}-P_i) > l(\mathscr{K}) - n$, then

$$\dim L\big(\mathscr{K}-\Sigma(P_i+\pi P_i)\big) = \dim \bigcap_1^g L(\mathscr{K}-P_i) > l(\mathscr{K}) - g = 0\,.$$

This shows that there is an effective divisor $E \sim \mathscr{K} - \Sigma(P_i+\pi P_i)$; but this is impossible since $\deg\big(\mathscr{K}-\Sigma(P_i+\pi P_i)\big) < 0$. □

COROLLARY. *Let* $\{P_1, P_3, ..., P_n\}$ *be disjoint. Then*

$$\dim\Big(L(\mathscr{K}-2P_1) \cap \bigcap_3^n L(\mathscr{K}-P_i)\Big) = g - n\,.$$

Proof. Since $L(\mathscr{K}-2P_1) \subset L(\mathscr{K}-P_1)$, by the lemma $L(\mathscr{K}-2P_1) \cap \bigcap_3^n L(\mathscr{K}-P_i)$ is contained in the vector space $L(\mathscr{K}-P_1) \cap \bigcap_3^n L(\mathscr{K}-P_i)$ of dimension $g - n + 1$.

If these vector spaces were equal, then they would both equal

$$L(\mathscr{K}-2P_1-\pi P_1) \cap \bigcap_3^n L(\mathscr{K}-P_i-\pi P_i)\,.$$

Choosing more points $P_{n+1}, ..., P_g$ as in the proof of the lemma would give, similarly,

$$\dim L(\mathscr{K}-2P_1-\pi P_1) \cap \bigcap_3^g L(\mathscr{K}-P_i-\pi P_i) \geqslant 1 .$$

Again, we get a contradiction since this shows that the divisor $\mathscr{K} - 2P_1 - \pi P_1 - \sum_3^g (P_i - \pi P_i)$ of negative degree is linearly equivalent to an effective divisor. □

Now I can finally prove (3).

Proof of (3). Given an effective special divisor D of degree $2r$ and with $\dim |D| = r$, choose points $P_1, ..., P_r$ forming a disjoint set. Notice that since $2 \leqslant \deg D$ and $2 \leqslant \deg(\mathscr{K}-D)$, then $1 \leqslant r \leqslant g - 2$. Then there is a divisor, call it D, in $|D|$ of the form

$$D = P_1 + ... + P_r + A .$$

I claim $A = \pi P_1 + ... + \pi P_r$. This could fail in two ways.

Case 1: If A contains some point Q which is not equal to any of $P_1, ..., P_r$ or $\pi P_1, ..., \pi P_r$, then $L(\mathscr{K}-D) \subset \bigcap_1^r L(\mathscr{K}-P_i) \cap L(\mathscr{K}-Q)$. Yet $l(\mathscr{K}-D) = \dim |\mathscr{K} - D| + 1 = g - r$ while, by Lemma 3, the intersection has dimension $g - (r+1)$. This shows that Case 1 cannot occur.

Case 2: If A contains some P_i, or contains some πP_i twice, (after interchanging P_i and πP_i if necessary and renumbering) we can write

$$D = 2P_1 + P_2 + ... + P_r + B$$

where B is effective, of degree $r - 1$. Here, $L(\mathscr{K}-D) \subset L(\mathscr{K}-2P_1) \cap \bigcap_2^r L(\mathscr{K}-P_i)$. Again, $l(\mathscr{K}-D) = g - r$, and by the corollary the dimension of the intersection is $g - (r+1)$. Case 2 cannot occur either.

Thus, $D \sim P_1 + ... + P_r + \pi P_1 + ... + \pi P_r$ so $D \sim r \cdot g_2^1$. In particular, if D is any divisor on C of degree 2 with $\dim |D| = 1$, D is linearly equivalent to a divisor in the given g_2^1. Thus a hyperelliptic curve has a unique g_2^1. □

It is interesting to compare the results of Clifford's theorem with those of the Riemann-Roch theorem, for hyperelliptic curves. Clifford's theorem shows that any special effective divisor D with $\dim |D| = \frac{1}{2} \deg D$ is linearly equivalent to a multiple of the unique g_2^1. In particular, for the canonical divisor $\mathscr{K}$ we have $\mathscr{K} \sim (g-1) \cdot g_2^1$. Conversely, the Riemann-Roch theorem shows that any divisor $D \sim r \cdot g_2^1$, where $1 \leqslant r \leqslant g - 1$, satisfies $\dim | D | = \frac{1}{2} \deg D$. To see this, note that the proof of part (3) shows that if $D \sim r \cdot g_2^1$ I can write

$$D \sim (P_1 + \pi P_1) + (P_2 + \pi P_2) + \ldots + (P_r + \pi P_r)$$

for a disjoint set of points $\{P_1, \ldots, P_r\}$. Then

$$L(\mathscr{K} - D) = L\Big(\mathscr{K} - \sum_{i=1}^{r} (P_i + \pi P_i)\Big) = \bigcap_{1}^{r} L(\mathscr{K} - P_i) .$$

By lemma 3 this set has dimension $g - r$; in other words, $\dim | \mathscr{K} - D | = g - r - 1 = \frac{1}{2} \deg (\mathscr{K} - D)$. By lemma 1, $\dim | D | = \frac{1}{2} \deg D$.

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