3. Clifford's Theorem — The elementary proof

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THEOREM (Riemann-Roch). *Let C be a projective nonsingular algebraic curve. The genus of C is a nonnegative integer g. For all divisors D on C,*

$$\dim |D| \geqslant \deg D - g .$$

If the strict inequality holds, D is special. *For all special divisors D,*

$$\dim |D| = \deg D + 1 - g + \dim |\mathscr{K} - D| .$$

COROLLARY. $\deg \mathscr{K} = 2g - 2$; $\dim |\mathscr{K}| = g - 1$; *and all divisors D of degree $> 2g - 2$ are nonspecial.*

3. CLIFFORD'S THEOREM — THE ELEMENTARY PROOF

Clifford's Theorem complements Riemann-Roch by providing information about special divisors, which of necessity are of small degree. The theorem also gives a sufficient condition that the curve C is hyperelliptic. (The theorem owes its name to the appearance of its first part in [1].) The proof I give here is elementary; more typical modern proofs [e.g. 4, Ch. IV, section 5 and 3, Ch. 2, section 3] involve considering whether the canonical morphism $C \to \mathbf{P}_{g-1}$ defined by the canonical divisor $\mathscr{K}$ is an embedding.

Definition. C is a *hyperelliptic* curve if its genus g is at least 2, and if C admits a g^1_2.

Remarks.

1. C is hyperelliptic if and only if there is a rational map $C \to \mathbf{P}_1$ of degree 2.

2. This happens if and only if C has an (affine) equation of the form $y^2 = f(x)$.

3. Part (3) of Clifford's Theorem shows that a hyperelliptic curve has a unique g^1_2. Contrast this to the case of an elliptic curve, where $g = 1$. Here any divisor of degree 2 defines a g^1_2. Yet choosing distinct points P, Q one sees easily that the divisors $2P$ and $P + Q$ are not linearly equivalent, and so define distinct g^1_2's.

THEOREM (Clifford). *Let C be a curve of genus g, and let D be an effective special divisor on C. Then*

(1) $\dim |D| \leqslant \frac{1}{2} \deg D$.

(2) *Equality holds in only 3 cases:* (a) $D = 0$; *or*

(b) $D = \mathscr{K}$; *or*

(c) *C is a hyperelliptic curve.*

(3) *If Case 2c holds then C admits a unique g^1_2, $\deg D = 2r$ for some integer $r \geqslant 1$, and $D \sim r \cdot g^1_2$.*

Proof of (1). Since D is effective special, the vector spaces $L(D)$ and $L(\mathscr{K}-D)$ are both of positive dimension. Define a map $\mu: L(D) \times L(\mathscr{K}-D) \to L(\mathscr{K})$ by $\mu(f, g) = f \cdot g$. (Since $(f) + D > 0$ and $(g) + \mathscr{K} - D > 0$, $(fg) + \mathscr{K} = (f) + (g) + \mathscr{K} = [(f)+D] + [(g)+\mathscr{K}-D] > 0$ so $fg \in L(\mathscr{K})$.) This map is bi-injective, so $\dim L(\mathscr{K}) \geqslant \dim L(D) + \dim L(\mathscr{K}-D) - 1$ by Clifford's Lemma. Since $l(D) = \dim |D| - 1$, one has

$$(1) \qquad \dim |\mathscr{K}| \geqslant \dim |D| + \dim |\mathscr{K} - D| .$$

On the other hand, Riemann-Roch guarantees that

$$(2) \qquad \deg D + 1 - g = \dim |D| - \dim |\mathscr{K} - D| .$$

Adding these, and recalling that $\dim |\mathscr{K}| = g - 1$, one gets $\deg D \geqslant 2 \dim |D|$. □

Implicit in the proof is a result I will need later.

LEMMA 1. *For the effective special divisor D, $\dim |D| = \frac{1}{2} \deg D$ if and only if $\dim |\mathscr{K}| = \dim |D| + \dim |\mathscr{K} - D|$. This holds if and only if $g - 1 \leqslant \dim |D| + \dim |\mathscr{K} - D|$. Further, equality holds for D if and only if it holds for (any effective divisor linearly equivalent to) $\mathscr{K} - D$.* □

Proof of (2). Assume that equality holds, and that D is neither 0 nor $\mathscr{K}$. Notice that if $\deg D = 2$, or $\deg \mathscr{K} - D = 2$, then D, or $\mathscr{K} - D$, defines a g^1_2 and C is hyperelliptic. Thus, I may assume that $\deg D$ and $\deg \mathscr{K} - D$ are both at least 4, so $\dim |D|$ and $\dim |\mathscr{K} - D|$ are both at least 2. Fix a point P in C. Since $\dim |\mathscr{K} - D| \geqslant 2$ I can choose a divisor $E = P + \Sigma e_R R$ in $|\mathscr{K} - D|$. Now fix a point Q on C but not in the support of E (i.e. $e_Q = 0$). Because $\dim |D| \geqslant 2$ I can choose a divisor (sloppily I call it D) in $|D|$ whose support contains both P and Q,

$$D = P + Q + \Sigma d_R R .$$

Set $I = \inf(D, E)$ and $S = \sup(D, E)$. Then

$$I = \Sigma \min(d_P, e_P) \cdot P \quad \text{and} \quad S = \Sigma \max(d_P, e_P) \cdot P .$$

Since P is in I, and Q is not, we have $0 < \deg I < \deg D$. Once I show that $\dim |I| = \frac{1}{2} \deg I$, by descent I will have shown that C is hyperelliptic.

Notice that $L(I) = L(D) \bigcap L(E)$. The inclusion $L(I) \subset L(D) \bigcap L(D)$ holds because $I < D$ and $I < E$. On the other hand, if $f \in L(D) \bigcap L(E)$, $(f) + D$ and $(f) + E$ are both effective. Then, for all points R, $\operatorname{ord}_R(f) \geqslant -d_R$ and $\operatorname{ord}_R(f) \geqslant -e_R$, so $\operatorname{ord}_R(f) + \min(d_R, e_R) \geqslant 0$ and $f \in L(I)$. Similarly, one sees that $L(D) + L(E) \subset L(S)$. Since $D < S$ and $E < S$ both $L(D)$ and $L(E)$ are subspaces of $L(S)$. If $\delta \in L(D)$ and $\varepsilon \in L(E)$, then for all R, $\operatorname{ord}_R(\delta+\varepsilon) \geqslant \min(\operatorname{ord}_R(\delta), \operatorname{ord}_R(\varepsilon)) \geqslant \min(-d_R, -e_R) = -\max(d_R, e_R)$. This shows that $\delta + \varepsilon \in L(S)$.

As subspaces of $L(S)$, we see that

$$\dim L(D) + \dim L(E) = \dim L(I) + \dim (L(D) + L(E)) .$$

Rewriting this in terms of linear systems gives

$$\dim |D| + \dim |E| \leqslant \dim |I| + \dim |S| .$$

Since $E \sim \mathscr{K} - D$, Lemma 1 applied to D gives

$$\dim |\mathscr{K}| \leqslant \dim |I| + \dim |S| .$$

Yet $I + S = D + E \sim \mathscr{K}$, so $S \sim \mathscr{K} - I$. Lemma 1, now applied to I, shows that $\dim |I| = \frac{1}{2} \deg I$. □

To prove the third part of the theorem I need some technical lemmas. We may assume that the curve C is hyperelliptic and so comes equipped with a given g^1_2. On any such curve I can define a function $\pi: C \to C$, by defining $\pi(P)$ to be the unique point Q such that $P + Q$ is a divisor in the given g^1_2. To verify that $\pi(P)$ is well defined, notice that if $P + Q$ and $P + R$ both belong to the given g^1_2, then $Q \sim R$. Since $g > 0$, Q must equal R [4, II. 6.10.1]; this shows that $\pi(P)$ is well-defined. Notice that since $\pi P + P$ is in the g^1_2, $\pi(\pi P) = P$.

Lemma 2. *For any point* P, $L(\mathscr{K} - P) = L(\mathscr{K} - P - \pi P)$ *and* $l(\mathscr{K} - P) = l(\mathscr{K}) - 1$.

Proof. $P + \pi(P)$ is a g^1_2 so $\dim |P + \pi P| = 1$ and by Lemma 1, $1 + \dim |\mathscr{K} - P - \pi P| = \dim |\mathscr{K}|$. Since $\mathscr{K} - P - \pi P < \mathscr{K} - P$

$< \mathcal{K}$, one sees that $L(\mathcal{K}-P-\pi P) \subset L(\mathcal{K}-P) \subset L(\mathcal{K})$. To prove $L(\mathcal{K}-P) = L(\mathcal{K}-P-\pi P)$ it suffices to show that $L(\mathcal{K}-P) \neq L(\mathcal{K})$. Yet if these were equal, the divisor P would be an effective special divisor of degree 1 with $\dim|\mathcal{K}-P| = \dim|\mathcal{K}|$. By Lemma 1, then $\dim|P|$ would equal $\frac{1}{2}\deg P$, which is absurd! □

Definition. The points $P_1, \ldots, P_k$ on C form a *disjoint set of points* if for each i, $P_i \neq \pi(P_i)$ and if the divisors $P_i + \pi P_i$ are pairwise disjoint.

LEMMA 3. *Let* $\{P_1, \ldots, P_n\}$ *be a disjoint set of points, with* $n \leqslant g$. *Then*

$$\dim \bigcap_1^n L(\mathcal{K}-P_i) = l(\mathcal{K}) - n = g - n\,.$$

Proof. Since $l(\mathcal{K}-P_i) = l(\mathcal{K}) - 1$, the intersection has dimension $\geqslant l(\mathcal{K}) - n$. Choose points $P_{n+1}, \ldots, P_g$ such that $\{P_1, \ldots, P_g\}$ is a disjoint set. Then

$$\bigcap_1^g L(\mathcal{K}-P_i) = \bigcap_1^g L(\mathcal{K}-P_i-\pi P_i) = L\Big(\mathcal{K}-\sum_1^g (P_i+\pi P_i)\Big)\,.$$

If $\dim \bigcap_1^n L(\mathcal{K}-P_i) > l(\mathcal{K}) - n$, then

$$\dim L\big(\mathcal{K}-\Sigma(P_i+\pi P_i)\big) = \dim \bigcap_1^g L(\mathcal{K}-P_i) > l(\mathcal{K}) - g = 0\,.$$

This shows that there is an effective divisor $E \sim \mathcal{K} - \Sigma(P_i+\pi P_i)$; but this is impossible since $\deg\big(\mathcal{K}-\Sigma(P_i+\pi P_i)\big) < 0$. □

COROLLARY. *Let* $\{P_1, P_3, \ldots, P_n\}$ *be disjoint. Then*

$$\dim\Big(L(\mathcal{K}-2P_1) \cap \bigcap_3^n L(\mathcal{K}-P_i)\Big) = g - n\,.$$

Proof. Since $L(\mathcal{K}-2P_1) \subset L(\mathcal{K}-P_1)$, by the lemma $L(\mathcal{K}-2P_1) \cap \bigcap_3^n L(\mathcal{K}-P_i)$ is contained in the vector space $L(\mathcal{K}-P_1) \cap \bigcap_3^n L(\mathcal{K}-P_i)$ of dimension $g - n + 1$.

If these vector spaces were equal, then they would both equal

$$L(\mathcal{K}-2P_1-\pi P_1) \cap \bigcap_3^n L(\mathcal{K}-P_i-\pi P_i)\,.$$

Choosing more points $P_{n+1}, ..., P_g$ as in the proof of the lemma would give, similarly,

$$\dim L(\mathscr{K}-2P_1-\pi P_1) \cap \bigcap_3^g L(\mathscr{K}-P_i-\pi P_i) \geqslant 1 .$$

Again, we get a contradiction since this shows that the divisor $\mathscr{K} - 2P_1 - \pi P_1 - \sum_3^g (P_i-\pi P_i)$ of negative degree is linearly equivalent to an effective divisor. □

Now I can finally prove (3).

Proof of (3). Given an effective special divisor D of degree $2r$ and with $\dim | D | = r$, choose points $P_1, ..., P_r$ forming a disjoint set. Notice that since $2 \leqslant \deg D$ and $2 \leqslant \deg(\mathscr{K}-D)$, then $1 \leqslant r \leqslant g - 2$. Then there is a divisor, call it D, in $| D |$ of the form

$$D = P_1 + ... + P_r + A .$$

I claim $A = \pi P_1 + ... + \pi P_r$. This could fail in two ways.

Case 1: If A contains some point Q which is not equal to any of $P_1, ..., P_r$ or $\pi P_1, ..., \pi P_r$, then $L(\mathscr{K}-D) \subset \bigcap_1^r L(\mathscr{K}-P_i) \cap L(\mathscr{K}-Q)$. Yet $l(\mathscr{K}-D) = \dim | \mathscr{K} - D | + 1 = g - r$ while, by Lemma 3, the intersection has dimension $g - (r+1)$. This shows that Case 1 cannot occur.

Case 2: If A contains some P_i, or contains some πP_i twice, (after interchanging P_i and πP_i if necessary and renumbering) we can write

$$D = 2P_1 + P_2 + ... + P_r + B$$

where B is effective, of degree $r - 1$. Here, $L(\mathscr{K}-D) \subset L(\mathscr{K}-2P_1) \cap \bigcap_2^r L(\mathscr{K}-P_i)$. Again, $l(\mathscr{K}-D) = g - r$, and by the corollary the dimension of the intersection is $g - (r+1)$. Case 2 cannot occur either.

Thus, $D \sim P_1 + ... + P_r + \pi P_1 + ... + \pi P_r$ so $D \sim r \cdot g_2^1$. In particular, if D is any divisor on C of degree 2 with $\dim | D | = 1$, D is linearly equivalent to a divisor in the given g_2^1. Thus a hyperelliptic curve has a unique g_2^1. □

It is interesting to compare the results of Clifford's theorem with those of the Riemann-Roch theorem, for hyperelliptic curves. Clifford's theorem shows that any special effective divisor D with $\dim | D | = \frac{1}{2} \deg D$ is linearly equivalent to a multiple of the unique g^1_2. In particular, for the canonical divisor $\mathscr{K}$ we have $\mathscr{K} \sim (g-1) \cdot g^1_2$. Conversely, the Riemann-Roch theorem shows that any divisor $D \sim r \cdot g^1_2$, where $1 \leqslant r \leqslant g - 1$, satisfies $\dim | D | = \frac{1}{2} \deg D$. To see this, note that the proof of part (3) shows that if $D \sim r \cdot g^1_2$ I can write

$$D \sim (P_1 + \pi P_1) + (P_2 + \pi P_2) + \ldots + (P_r + \pi P_r)$$

for a disjoint set of points $\{P_1, \ldots, P_r\}$. Then

$$L(\mathscr{K} - D) = L\Big(\mathscr{K} - \sum_{i=1}^{r} (P_i + \pi P_i)\Big) = \bigcap_{1}^{r} L(\mathscr{K} - P_i)\,.$$

By lemma 3 this set has dimension $g - r$; in other words, $\dim | \mathscr{K} - D | = g - r - 1 = \frac{1}{2} \deg (\mathscr{K} - D)$. By lemma 1, $\dim | D | = \frac{1}{2} \deg D$.

REFERENCES

[1] Clifford, William Kingdon. On the Classification of Loci. XXXIII in *Collected Papers*, London, Macmillan & Co., 1882.

[2] Fulton, William. *Algebraic Curves*. Reading, MA, Addison-Wesley, 1969.

[3] Griffiths, Philipp and Joseph Harris. *Principles of Algebraic Geometry*. New York, John Wiley & Sons, 1978.

[4] Hartshorne, Robin. *Algebraic Geometry*. New York, Springer-Verlag, 1977.

[5] Walker, Robert J. *Algebraic Curves*. New York, Dover, 1962.

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