§1. The Bergman-Roseblade Theorem

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§ 1. The Bergman-Roseblade Theorem

We introduce some notation. Let A denote a finitely generated free abelian group. Let k be a field and have $\underline{A}$ designate an isomorphic copy of A inside the multiplicative group of some field extension of k. We will write $k[\underline{A}]$ for the k-algebra generated by $\underline{A}$ and $k(\underline{A})$ for its field of fractions. The reader is cautioned that $k[\underline{A}]$ is not the group algebra; distinct elements of $\underline{A}$ need not be linearly independent over k. (It's even possible for $\underline{A}$ to be contained in k^*.)

We will reserve the notation $k[A]$, without the underbar, for the group algebra. There is an obvious relation between the free object $k[A]$ and $k[\underline{A}]$. Indeed, the given isomorphism $A \simeq \underline{A}$ induces a k-algebra isomorphism $k[A]/P \simeq k[\underline{A}]$ where P is a prime ideal. The ideal P is "faithful with respect to A."

Suppose that G is a group which acts on A. Set

$$D = \{a \in A \mid a \text{ has a finite } G\text{-orbit}\}.$$

It is sometimes called the orbital subgroup or relative finite conjugate subgroup.

We are primarily interested in a group G which acts as a group of k-automorphisms of $k[\underline{A}]$. (The slight awkwardness of language allows us to include possibly nonfaithful actions.) We say that G *acts multiplicatively* on $k[\underline{A}]$ if G stabilizes $\underline{A}$. Thus if $k[\underline{A}] = k[A]/P$ as described above, then P is a G-stable ideal under the corresponding action on $k[A]$.

The fundamental theorem in multiplicative invariant theory is Roseblade's Theorem D ([10]). Roseblade based his arguments on profound insights of G. Bergman ([3]).

Bergman-Roseblade Theorem. *Assume that* G *acts multiplicatively on* $k[\underline{A}] = k[A]/P$. *Then* $P = (P \cap k[D])k[A]$.

To understand the implications of this theorem, we take a closer look at D. Obviously D is a finitely generated abelian group. Since each generator has a finite G-orbit, D is centralized by a subgroup of finite index in G. In other words, G acts like a finite group of automorphisms when restricted to D.

It is easy to see that if a power of an element in A has a finite orbit, then so does the original element. Hence there is a splitting $A = D \times B$. (Unfortunately, there may be no choice of B which is stabilized by G.) The conclusion of the Bergman-Roseblade Theorem can be rewritten—every element in P has a unique representation $\Sigma f(b)b$ where $b \in B$ and $f(b) \in P \cap k[D]$. Thus $k[\underline{A}]$ is the group ring $(k[\underline{D}])[B]$ for a finitely generated free abelian group B.

Roseblade proves that the fixed ring $(k[\underline{A}])^G$ lies in $k[\underline{D}]$ ([10], Lemma 10). This will also be a consequence of the first lemma in the next section. In any event, it has a remarkable consequence.

THEOREM 1. *Assume that* G *is an arbitrary group acting multiplicatively on* $k[\underline{A}]$. *Then* $k[\underline{A}]^G$ *is finitely generated.*

Proof. As we have remarked, $(k[\underline{A}])^G = (k[\underline{D}])^G$. But G acts like a finite group of automorphisms on the affine algebra $k[\underline{D}]$. Noether's Theorem ([11]), states that, in this case, the algebra of invariants is a finitely generated algebra. □

This is an unexpected surprise. In contrast to the situation for linear actions, Hilbert's 14th problem holds for multiplicative actions without any restriction on the group!

The theme of the paper has emerged. A theory of invariants for multiplicative actions is ultimately a theory for finite groups.

§ 2. GALOIS THEORY

We begin this section by establishing an analogue to the "finiteness" phenomenon of the previous section, for a multiplicative action of G on $k(\underline{A})$. Notation is taken from § 1.

LEMMA 2. *Suppose that* G *acts multiplicatively on* $k(\underline{A})$. *Then* $k(\underline{A})^G \subset k(\underline{D})$.

Proof. The crucial fact is that $k(\underline{D})[B]$ is a unique factorization domain. If ${}^g f = f$ for $f \in k(\underline{A})$ then we can write $f = \alpha/\beta$ where α and β in $k(\underline{D})[B]$ have no common factors. The invariance of f becomes

$$({}^g\alpha)\beta = ({}^g\beta)\alpha \quad \text{for all} \quad g \in G.$$

Hence $\alpha \mid {}^g\alpha$ and ${}^g\alpha \mid \alpha$; we have $({}^g\alpha)\alpha^{-1}$ a unit in $k(\underline{D})[B]$. A similar result holds for β.

$${}^g\alpha = u(g)\alpha \quad \text{and} \quad {}^g\beta = w(g)\beta$$

for $u(g), w(g) \in k(\underline{D})^* \cdot B$.