§3. The Shepherd-Todd-Chevalley Theorem

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$$k(\underline{A})^G_\lambda = (k[\underline{A}]^G)^{-1}(k[\underline{A}]^G_\lambda).$$

Recall that $G/C_G(D)$ is a finite group. Hence $\mathrm{Hom}(G/C_G(D), k^*)$ is finite. Consequently, when $\mathrm{Hom}(G, k^*)$ is infinite the proposition implies that $H^1(G, k(\underline{A})^*) \neq 1$. It is quite plausible (under the assumption $k^* \cap \underline{A} = 1$) that $H^1(G, k(\underline{A})^*)$ vanishes if and only if G is finite.

The extra bothersome assumption is vacuous in the case of group algebras. One can read off the following observation from Lemma 2′.

PROPOSITION 6. *Assume that* $D = 1$. *Then*

$$1 \to \mathrm{Hom}(G, k^*) \times H^1(G, A) \to H^1(G, k(A)^*) \quad \textit{is exact}. \qquad \square$$

I have been unable to determine if the injection given by the proposition always splits. Here is one situation where it does.

PROPOSITION 7. *Suppose that* A *can be fully ordered so that* G *acts as a group of order automorphisms of* A. *Then the natural map*

$$H^1(G, k^* \cdot A) \to H^1(G, k(A)^*)$$

splits.

Proof. Let $V: k[A]\backslash\{0\} \to k^* \cdot A$ be the function which sends an element to its "lowest term" with respect to the ordering. The usual degree argument which shows that a polynomial ring is a domain, establishes that V is multiplicative. Since elements of G act monotonically, V is a map of (multiplicative) G-modules. It is not difficult to check that V extends to a multiplicative G-map from $k(A)^*$ to $k^* \cdot A$.

Obviously $k^* \cdot A \to k(A)^* \xrightarrow{V} k^* \cdot A$ provides the necessary splitting. $\square$

The hypothesis of Proposition 7 is very restrictive, even for an infinite cyclic group G. We leave the following long exercise to the reader. A matrix in $GL(n, \mathbf{Z})$ is order preserving for some ordering on $\mathbf{Z}^n$ and only if each rational irreducible factor of its characteristic polynomial has a positive real root.

§ 3. THE SHEPHARD-TODD-CHEVALLEY THEOREM

Recall that a matrix in $GL(n, \mathbf{C})$ is a pseudo-reflection if it has finite order and 1 is an eigenvalue of multiplicity $n - 1$. The remaining eigenvalue for a pseudo-reflection must be a root of unity; when it is -1 we call the matrix a reflection. Notice that every pseudo-reflection in $GL(n, \mathbf{Z})$ must be a reflection. A pseudo-reflection group (resp. reflection group) is a finite group generated by pseudo-reflections (resp. reflections). The classical result is the

SHEPHARD-TODD-CHEVALLEY THEOREM (cf. [11], Theorem 4.2.5). *Suppose that G is a finite group of automorphisms of $\mathbf{C}[X_1, ..., X_n]$ which acts linearly. Then $\mathbf{C}[X_1, ..., X_n]^G$ is a polynomial ring if and only if G is a pseudo-reflection group.*

The major theorem of this section is one direction of the STC Theorem for multiplicative actions. Namely,

THEOREM 8. *Suppose that $G \subset GL(n, \mathbf{Z})$ is a finite group of automorphisms of $A \simeq \mathbf{Z}^n$. If $\mathbf{C}[A]^G$ is a polynomial ring then G is a reflection group.*

This theorem is deduced from the STC Theorem via a connection between abelian group algebras and polynomial rings which goes back to the pioneers of infinite group theory. From now on A will be the free abelian group on n generators. Let V be the n-dimensional complex vector space $\mathbf{C} \otimes_{\mathbf{Z}} A$. If x is in A we shall write $\bar{x} = 1 \otimes x$ in V. The symmetric algebra on V will be denoted $\mathbf{C}[V]$. (We warn the reader of our primitive tendencies; $\mathbf{C}[V]$ is not the algebra of polynomial functions on V.) Both $\mathbf{C}[A]$ and $\mathbf{C}[V]$ have canonical augmentations. In the former case the augmentation ideal ω is the ideal generated by $\{x - 1 \mid x \in A\}$. In the latter, ω is the ideal generated by vectors in V. Let $\mathbf{C}[A]^\wedge$ and $\mathbf{C}[V]^\wedge$ be the respective ω-adic completions. The exponential function from A into $\mathbf{C}[V]^\wedge$ given by

$$\exp(x) = \sum_{j=0}^{\infty} (\bar{x}^j)/j!$$

is well-defined. It extends by linearity and then continuity to a $\mathbf{C}$-algebra map E: $\mathbf{C}[A]^\wedge \to \mathbf{C}[V]^\wedge$. In fact, E is an isomorphism. (The map back extends the logarithm.)

The effect of this identification on automorphisms was first exploited in [1]. A matrix $g \in GL(A)$ induces an automorphism γ on $\mathbf{C}[A]^\wedge$. What is the automorphism after "translating" by E? The following calculation of $E\gamma E^{-1}$ on x can be checked in detail on the matrix level:

$$(E\gamma E^{-1})(\bar{x}) = E\gamma E^{-1}(\log E(x)) = E(\log^g x)$$
$$= E(g(\log x)) = g(\log E(x)) = g(\bar{x}).$$

LINEARIZATION THEOREM. *Let* G *be a group of automorphisms of* A, *regarded in* $GL(n, \mathbf{Z})$. *Exponentiation extends to an algebra isomorphism* $E: \mathbf{C}[A]^{\wedge} \to \mathbf{C}[V]^{\wedge}$. *Moreover, the multiplicative action of* G *(extended by continuity) on* $\mathbf{C}[A]^{\wedge}$ *induces an action on* $\mathbf{C}[V]^{\wedge}$ *which is the extension (by continuity) of the linear action of* G *on* $\mathbf{C}[V]$. □

With this tool in hand, the proof of Theorem 8 amounts to carefully keeping track of a myriad of completions and then getting rid of them. The calculations are somewhat clearer in the abstract. So let S be a C-algebra and let G be a finite group of automorphisms of S. The averaging or Reynolds operator which sends S to the fixed ring S^G is given by

$$\mathrm{av}(c) = \frac{1}{|G|} \sum_{g \in G} {}^{g}c$$

The function av is an idempotent S^G-module map.

LEMMA 9. *Suppose that* S *is a commutative noetherian* $\mathbf{C}$*-algebra and* I *is a G-stable maximal ideal. Then there is a positive integer* f *such that*

$$I^{ft} \cap S^G \subset (I \cap S^G)^t \subset I^t \cap S^G \quad \textit{for} \quad t = 1, 2, \ldots .$$

Proof. The second inclusion is obvious. Set $J = I \cap S^G$. We first prove that I is the only prime ideal lying over JS.

Indeed, suppose P is a prime ideal of S containing J. If $a \in I$ then $\prod_{g \in G} {}^{g}a \in I \cap S^G \subset P$. By primality there is some $g \in G$ with $a \in {}^{g}P \cap I$. Consequently, $I = \bigcup_{g \in G} ({}^{g}P \cap I)$, a union of complex subspaces. At least one of these subspaces is not proper: there is an $h \in G$ such that $I = {}^{h}P \cap I$. Therefore $I = {}^{h^{-1}}I \subset P$. Maximality implies $I = P$, as required.

The prime radical of S/JS is the image of I. But the prime radical is nil and nil ideals in a noetherian ring are nilpotent. Hence there is a positive integer f such that $I^f \subset JS$.

We have established, so far, that $I^{ft} \subset J^t S$ for all t. Intersect each side of the inclusion S^G and apply the averaging operator.

$$I^{ft} \cap S^G = \mathrm{av}(I^{ft} \cap S^G) = \mathrm{av}(J^t S \cap S^G) \subset \mathrm{av}(J^t S) = J^t \mathrm{av}(S)$$

We have obtained the necessary inclusion:

$$I^{ft} \cap S^G \subset J^t = (I \cap S^G)^t .$$ □

LEMMA 10. *Suppose that* S *has a filtration* $S = S_0 \supset S_1 \supset S_2 \supset \ldots$ *such that each* S_j *is G-stable and* $\cap S_j = 0$. *Then* $(S^{\wedge})^G = (S^G)^{\wedge}$. (Here

$S^\wedge$ denotes the completion of S with respect to the given filtration and $(S^G)^\wedge$ means the completion of S^G for the "relative" filtration $S_j \cap S^G$.)

Proof. There is an obvious injection $(S^G)^\wedge \to S^\wedge$, where the topology on $(S^G)^\wedge$ coincides with the relative topology on its image. Notice that the action of G on S extends continuously to an action on $S^\wedge$: if $a_m \to a$ then ${}^g a_m \to {}^g a$. It follows that $(S^G)^\wedge \subset (S^\wedge)^G$.

Suppose $b \in (S^\wedge)^G$. Choose a sequence $b_m \in S$ such that $b_m \to b$. Then $\mathrm{av}(b_m) \to \mathrm{av}(b)$ and $\mathrm{av}(b) = b$. Hence $b \in (S^G)^\wedge$. □

LEMMA 11. *Suppose that k is a field and $\Phi: k[T_1, ..., T_n] \to k$ is a k-algebra homomorphism. Then there is a change of variables,*

$$k[T_1, ..., T_n] = k[T'_1, ..., T'_n],$$

so that $\ker \Phi = (T'_1, ..., T'_n)$.

Proof. Consider the automorphism induced by sending each T_j to $T'_j = T_j - \Phi(T_j)$. □

The next lemma is undoubtedly routine for the expert in commutative algebra. Rather than interrupt the flow of the narrative, we will state it now and then relegate a sketchy proof to the appendix.

DECOMPLETION LEMMA. *Let k be a field and suppose $R = R_{(0)} \oplus R_{(1)} \oplus \dots$ is a graded k-algebra with $R_{(0)} = k$. If $\hat{R}$ (its completion with respect to the grade filtration) is algebra isomorphic to a power series ring $k[[T_1, ..., T_n]]$ then R is isomorphic to a polynomial ring in n homogeneous variables.*

Proof of Theorem 8 that if $\mathbf{C}[A]^G$ is a polynomial ring then G is a reflection group: According to Lemma 10, $(\mathbf{C}[A]^\wedge)^G = (\mathbf{C}[A]^G)^\wedge$. Here $(\mathbf{C}[A]^G)^\wedge$ is the completion of $\mathbf{C}[A]^G$ with respect to the filtration $\omega^t \cap \mathbf{C}[A]^G$. A straightforward Cauchy sequence argument in conjunction with Lemma 9 shows that $(\mathbf{C}[A]^G)^\wedge$ is also the $(\omega \cap \mathbf{C}[A]^G)$-adic completion. Now $\mathbf{C}[A]^G$ is a polynomial ring in $n = \mathrm{rank} A$ variables and $\omega \cap \mathbf{C}[A]^G$ is a codimension one ideal. By Lemma 11, the $(\omega \cap \mathbf{C}[A]^G)$-adic completion of $\mathbf{C}[A]^G$ is isomorphic to the power series ring $\mathbf{C}[[T_1, ..., T_n]]$.

In summary, $(\mathbf{C}[A]^\wedge)^G \simeq \mathbf{C}[[T_1, ..., T_n]]$. Next, apply the isomorphism E and use Lemma 10 for the symmetric algebra. We find that $(\mathbf{C}[V]^G)^\wedge \simeq \mathbf{C}[[T_1, ..., T_n]]$. This time, $\mathbf{C}[V]^G$ is a *graded* algebra under the grading inherited from $\mathbf{C}[V]$ and its completion is with respect to the grade filtration.

We are in the situation of the Decompletion Lemma for $\mathbf{C}[V]^G = R$. Thus $\mathbf{C}[V]^G$ is a polynomial ring in n homogeneous variables. Our theorem now follows from the STC Theorem. □

It is possible to object to the appropriateness of proving a theorem which determines when the invariants for a group algebra comprise a polynomial algebra. After all, the most well-behaved group is the group of order one and its fixed ring is the group algebra we began with. Let's say that a $\mathbf{C}$-algebra is an *extended polynomial ring* if it contains algebraically independent elements $U_1, \ldots, U_m, T_1, \ldots, T_n$ such that the algebra is isomorphic to $\mathbf{C}[U_1, U_1^{-1}, \ldots, U_m, U_m^{-1}, T_1, \ldots, T_n]$. Equivalently, an extended polynomial ring has the form $\mathbf{C}[U] \otimes_{\mathbf{C}} \mathbf{C}[T_1, \ldots, T_n]$ where U is a finitely generated free abelian group. Once the generators U_i and T_j are distinguished, its augmentation ideal ω is the ideal generated by $U_1 - 1, \ldots, U_m - 1, T_1, \ldots, T_n$.

The theorem we have proved can be adapted to prove the "correct" result.

THEOREM 8^+. *Suppose G is a finite group acting faithfully and multiplicatively on $\mathbf{C}[A]$. If $\mathbf{C}[A]^G$ is an extended polynomial ring then G is a reflection group.*

Proof. We follow the argument a few lines up. It is still true that $(\mathbf{C}[A]^\wedge)^G$ is the $(\omega \cap \mathbf{C}[A]^G)$-adic completion of $\mathbf{C}[A]^G$. This time $\omega \cap \mathbf{C}[A]^G$ is a codimension one ideal in the extended polynomial ring $\mathbf{C}[A]^G$. We need Lemma 11^+: if

$$\Phi: k[U_1^{\pm 1}, \ldots, U_m^{\pm 1}, T_1, \ldots, T_n] \to k$$

is an algebra homomorphism then there is a change of variables so that $\ker\Phi$ becomes the augmentation ideal. (Indeed, define $U'_j = \Phi(U_j)^{-1} U_j$ and $T'_j = T_j - \Phi(T_j)$.)

What is the completion of an extended polynomial ring with respect to powers of its augmentation ideal? Topological abstract nonsense shows that it coincides with $\mathbf{C}[U]^\wedge[[T_1, \ldots, T_n]]$ where $\mathbf{C}[U]^\wedge$ is the completion of the group algebra with respect to the $(U_1 - 1, \ldots, U_m - 1)$-adic topology. But the linearizing E-isomorphism exhibits $\mathbf{C}[U]^\wedge$ as a power series ring in $\operatorname{rank} U$ variables. In summary, the augmentation-adic completion of an extended polynomial ring is also a power series ring.

From here on, the previous argument can be carried over verbatim. □

It is much more difficult to decide when $\mathbf{C}(A)^G$ is a rational function field. The little that is known is surveyed in [7].