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**Autor:** Farkas, Daniel R.  
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## § 4. APPENDIX

$R = R_{(0)} \oplus R_{(1)} \oplus \dots$  is a graded  $k$ -algebra with  $R_{(0)} = k$ . Let  $\mathfrak{m}$  be the maximal ideal  $\sum_{i=1}^{\infty} R_{(i)}$ . We assume that  $\hat{R}$  is a power series ring in finitely many variables. Obviously  $\hat{\mathfrak{m}}$  corresponds to the unique maximal ideal of the power series ring, whence  $\hat{R}/\hat{\mathfrak{m}}^d$  is always finite dimensional. Since  $\hat{\mathfrak{m}}^d$  is homogeneous, some tail  $\prod_{i=d+1}^{\infty} R_{(i)}$  must then lie in  $\hat{\mathfrak{m}}^d$ . It follows that the graded algebra of  $R$  for the  $\mathfrak{m}$ -adic filtration is isomorphic to the graded algebra of  $\hat{R}$  for the  $\hat{\mathfrak{m}}$ -adic filtration. The power series assumption implies that the latter is simply a polynomial ring with the standard grading.

Clearly  $\mathfrak{m}^2 \subset \sum_{j=2}^{\infty} R_{(j)}$ . Hence  $R_{(1)}$  injects into  $\mathfrak{m}/\mathfrak{m}^2$ . Choose a basis for  $R_{(1)}$  over  $k$  and extend it to a list of homogeneous elements  $x_1, \dots, x_n$  in  $\mathfrak{m}$  whose images constitute a basis for  $\mathfrak{m}/\mathfrak{m}^2$ . It is generally true for any commutative  $k$ -algebra  $R$  that when  $R/\mathfrak{m} = k$  and when the associated graded ring for the  $\mathfrak{m}$ -adic filtration is the symmetric algebra on  $\mathfrak{m}/\mathfrak{m}^2$ , that any basis for  $\mathfrak{m}/\mathfrak{m}^2$  pulls back to a set of algebraically independent elements in  $R$ . In particular,  $x_1, \dots, x_n$  are algebraically independent.

We use the given grading on  $R$  to prove that  $R = k[x_1, \dots, x_n]$ . Vacuously,  $R_{(0)} \subset k[x_1, \dots, x_n]$ . We have chosen the  $x_i$  so that  $R_{(1)}$  lies in their span, so  $R_{(1)} \subset k[x_1, \dots, x_n]$ . Assume, inductively, that  $d \geq 1$  and  $R_{(s)} \subset k[x_1, \dots, x_n]$  for all  $s \leq d$ . If  $y \in R_{(d+1)}$  then

$$y = \sum \lambda_i x_i + \sum u_j v_j$$

for some  $\lambda_i \in k$  and  $u_j, v_j \in \mathfrak{m}$ . Without loss of generality  $u_j$  and  $v_j$  are homogeneous and all the  $x_i$  and  $u_j v_j$  which appear in the formula lie in  $\bigcup_{t=1}^{d+1} R_{(t)}$ .

This can only happen when  $u_j$  and  $v_j$  are in  $R_{(s)}$  for some  $s \leq d$ .

By induction,  $u_j$  and  $v_j$  are elements of  $k[x_1, \dots, x_n]$ . Therefore  $y \in k[x_1, \dots, x_n]$ .

## § 5. WEYL GROUPS

It seems to be part of the folklore for Lie theory that the converse of Theorem 8 fails to be true (cf. [4] VI§ 3 Ex. 2). Rather than being dead-ends, these examples serve as inspiration: the machinery of root systems will allow us to determine the correct necessary and sufficient conditions