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§ 4. APPENDIX

$R = R_{(0)} \oplus R_{(1)} \oplus \dots$ is a graded k -algebra with $R_{(0)} = k$. Let \mathfrak{m} be the maximal ideal $\sum_{i=1}^{\infty} R_{(i)}$. We assume that \hat{R} is a power series ring in finitely many variables. Obviously $\hat{\mathfrak{m}}$ corresponds to the unique maximal ideal of the power series ring, whence $\hat{R}/\hat{\mathfrak{m}}^d$ is always finite dimensional. Since $\hat{\mathfrak{m}}^d$ is homogeneous, some tail $\prod_{i=d}^{\infty} R_{(i)}$ must then lie in $\hat{\mathfrak{m}}^d$. It follows that the graded algebra of R for the \mathfrak{m} -adic filtration is isomorphic to the graded algebra of \hat{R} for the $\hat{\mathfrak{m}}$ -adic filtration. The power series assumption implies that the latter is simply a polynomial ring with the standard grading.

Clearly $\mathfrak{m}^2 \subset \sum_{j=2}^{\infty} R_{(j)}$. Hence $R_{(1)}$ injects into $\mathfrak{m}/\mathfrak{m}^2$. Choose a basis for $R_{(1)}$ over k and extend it to a list of homogeneous elements x_1, \dots, x_n in \mathfrak{m} whose images constitute a basis for $\mathfrak{m}/\mathfrak{m}^2$. It is generally true for any commutative k -algebra R that when $R/\mathfrak{m} = k$ and when the associated graded ring for the \mathfrak{m} -adic filtration is the symmetric algebra on $\mathfrak{m}/\mathfrak{m}^2$, that any basis for $\mathfrak{m}/\mathfrak{m}^2$ pulls back to a set of algebraically independent elements in R . In particular, x_1, \dots, x_n are algebraically independent.

We use the given grading on R to prove that $R = k[x_1, \dots, x_n]$. Vacuously, $R_{(0)} \subset k[x_1, \dots, x_n]$. We have chosen the x_i so that $R_{(1)}$ lies in their span, so $R_{(1)} \subset k[x_1, \dots, x_n]$. Assume, inductively, that $d \geq 1$ and $R_{(s)} \subset k[x_1, \dots, x_n]$ for all $s \leq d$. If $y \in R_{(d+1)}$ then

$$y = \sum \lambda_i x_i + \sum u_j v_j$$

for some $\lambda_i \in k$ and $u_j, v_j \in \mathfrak{m}$. Without loss of generality u_j and v_j are homogeneous and all the x_i and $u_j v_j$ which appear in the formula lie in \mathfrak{m}^{d+1} .

$\cup_{t=1} R_{(t)}$. This can only happen when u_j and v_j are in $R_{(s)}$ for some $s \leq d$.

By induction, u_j and v_j are elements of $k[x_1, \dots, x_n]$. Therefore $y \in k[x_1, \dots, x_n]$.

§ 5. WEYL GROUPS

It seems to be part of the folklore for Lie theory that the converse of Theorem 8 fails to be true (cf. [4] VI§ 3 Ex. 2). Rather than being dead-ends, these examples serve as inspiration: the machinery of root systems will allow us to determine the correct necessary and sufficient conditions

for a multiplicative Shephard-Todd-Chevalley analogue. For the most part, we will follow the notation in [8].

Suppose that V is an n -dimensional complex vector space and $G \subset GL(V)$. By a G -lattice we mean a lattice in V (of rank n) which is invariant under the action of G . The G -lattice A is effective if zero is the only element fixed by all members of G . Notice that A is effective if and only if the units of $\mathbb{C}[A]^G$ are precisely the nonzero elements of \mathbb{C} .

PROPOSITION 12. *Let A be an effective G -lattice. If G is a finite group generated by reflections then*

- (i) *there is a reduced root system Φ lying in A so that G is the Weyl group for Φ , and*
- (ii) *A (considered inside V) lies between the root lattice for Φ and the weight lattice.*

Proof. Endow V with an inner product which makes members of G orthogonal transformations. If σ is a reflection in G and $a \in A$ is such that $a \neq \sigma(a)$ then $a - \sigma(a) \neq 0$ and $\sigma(a - \sigma(a)) = -(a - \sigma(a))$. Thus $\{b \in A \mid \sigma(b) = -b\}$ is an infinite cyclic subgroup of A . Its two possible generators, a_σ and $-a_\sigma$, are the nonzero vectors of smallest length in A which are "reflected" by σ . It is not difficult to check that $\Phi = \{\pm a_\sigma \mid \sigma \text{ is a reflection in } G\}$ is a root system, whence G is its Weyl group. Moreover, if $x \in A$ and $\alpha = \pm a_\sigma \in \Phi$ then $\sigma(x) \in A$. Thus $x - \frac{2(x, \alpha)}{(\alpha, \alpha)} \alpha \in A$. Now $\frac{2(x, \alpha)}{(\alpha, \alpha)} \alpha \in A$ implies that $\frac{2(x, \alpha)}{(\alpha, \alpha)}$ is an integer. This is just the statement that x is a weight. □

Although we have "located" the effective G -lattices, there are still quite a few of them: every lattice between the root lattice and the weight lattice is invariant under G . On the positive side, it turns out that the group algebra of the weight lattice has well-behaved invariants.

THEOREM ([4], VI § 3.4). *Let G be a Weyl group and let Λ be its weight lattice. Then $\mathbb{C}[\Lambda]^G$ is a polynomial ring.* □

This theorem of Bourbaki can be generalized just enough to suggest its own converse. Fix a root system with base Δ . Let Λ_r and Λ denote the root lattice and weight respectively and let w_1, \dots, w_n be the fundamental

dominant weights. Then Λ^+ is the collection of dominant weights: the non-negative integer combinations of w_1, \dots, w_n . Write W for the Weyl group.

In [5], we introduced the notion of *stretched weight lattice* for a root system. It is a W -lattice lying between Λ_r and Λ which has a basis of the form $r_1 w_1, r_2 w_2, \dots, r_n w_n$ for positive integers r_1, \dots, r_n . A stretched weight lattice can always be built up from ordinary weight lattices and certain root lattices ([5]). More unexpectedly, we found an abstract characterization. Suppose G is a finite subgroup of $GL(n, \mathbb{Z})$; then the corresponding action on \mathbb{Z}^n has the non-negative "quadrant" as fundamental domain (in Bourbaki's strong sense) if and only if G is a Weyl group and \mathbb{Z}^n is isomorphic to a stretched weight lattice for G .

To talk about the group algebra $\mathbb{C}[\Lambda]$, we will have to switch from additive to multiplicative notation for elements of Λ . If we think of λ as a weight then λ^* will be its image in $\mathbb{C}[\Lambda]$, e.g. $(\lambda_1 - \lambda_2)^* = (\lambda_1^*)(\lambda_2^*)^{-1}$.

For $\lambda \in \Lambda$ we set $X(\lambda) = (\text{constant}) \cdot \text{av}(\lambda^*)$ where the normalizing constant is chosen so that each element of Λ appears with coefficient 0 or 1 in $X(\lambda)$. Using this notation, we state the appropriate form of Bourbaki's Theorem. (The proof carries over verbatim from [4].)

THEOREM 13. *If S is a stretched weight lattice with basis $r_1 w_1, \dots, r_n w_n$ then*

$$\mathbb{C}[S]^W = \mathbb{C}[X(r_1 w_1), \dots, X(r_n w_n)].$$

Moreover, $X(r_1 w_1), \dots, X(r_n w_n)$ are algebraically independent. □

We shall frequently use the consequence that $X(w_1), \dots, X(w_n)$ are irreducible elements of the unique factorization domain $\mathbb{C}[\Lambda]^W$.

For the rest of this paper, M will be a W -lattice with

$$\Lambda_r \subset M \subset \Lambda.$$

LEMMA 14. *Suppose $\lambda_1, \dots, \lambda_t$ are (not necessarily distinct) dominant weights. If $\lambda_1 + \dots + \lambda_t \in M$ then $(g_1 \cdot \lambda_1) + \dots + (g_t \cdot \lambda_t) \in M$ for all choices $g_1, \dots, g_t \in W$.*

Proof. For $\alpha \in \Delta$ let σ_α denote reflection in the hyperplane perpendicular to α . Then $\sigma_\alpha(\lambda_j) = \lambda_j - \langle \lambda_j, \alpha \rangle \alpha$. The definition of "weight" implies that $\langle \lambda_j, \alpha \rangle$ is an integer. Thus

$$\sigma_\alpha(\lambda_j) \equiv \lambda_j \pmod{\Lambda_r}$$

and so,

$$\sigma_\alpha(\lambda_j) \equiv \lambda_j \pmod{M}.$$

Now W is generated by $\{\sigma_\alpha \mid \alpha \in \Delta\}$. An easy induction on the length of $g \in W$ as a word in the generators yields

$$g(\lambda_j) \equiv \lambda_j \pmod{M}.$$

Hence

$$\sum_{j=1}^t g_j(\lambda_j) \equiv \sum_{j=1}^t \lambda_j \pmod{M}. \quad \square$$

LEMMA 15. Suppose $\lambda_1, \dots, \lambda_t$ are (not necessarily distinct) dominant weights. If $\lambda_1 + \dots + \lambda_t \in M$ then

$$X(\lambda_1)X(\lambda_2) \cdots X(\lambda_t) \in \mathbf{C}[M]^W.$$

Proof. A typical element of Λ in the support of $X(\lambda_1) \cdots X(\lambda_t)$ has the form $(g_1(\lambda_1) + \dots + g_t(\lambda_t))^*$ where $g_1, \dots, g_t \in W$. According to Lemma 14, $\Sigma g_j(\lambda_j) \in M$. Thus

$$X(\lambda_1)X(\lambda_2) \cdots X(\lambda_t) \in \mathbf{C}[M] \cap \mathbf{C}[\Lambda]^W. \quad \square$$

We say that an element $w \in M \cap \Lambda^+$ is *M-indecomposable* if it cannot be written as a sum of two nonzero elements of $M \cap \Lambda^+$. Clearly, every element of $M \cap \Lambda^+$ is a sum of *M-indecomposable* elements.

THEOREM 16. The following statements are equivalent:

- (i) M is a stretched weight lattice for W .
- (ii) $\mathbf{C}[M]^W$ is a polynomial ring.
- (iii) $\mathbf{C}[M]^W$ is a UFD.

Proof. (i) \Rightarrow (ii) is Theorem 13 and (ii) \Rightarrow (iii) is classical. Thus we assume that $\mathbf{C}[M]^W$ is a UFD and prove (i).

Suppose $\sum_{j=1}^n a_j w_j$ is *M-indecomposable*. According to Lemma 15,

$$Y = X(w_1)^{a_1} X(w_2)^{a_2} \cdots X(w_n)^{a_n}$$

is an element of $\mathbf{C}[M]^W$. Every coefficient appearing in $X(w_j)$ is 1; hence any subproduct

$$X(w_1)^{b_1} X(w_2)^{b_2} \cdots X(w_n)^{b_n}$$

with $0 \leq b_j \leq a_j$ contains $(\sum_{j=1}^n b_j w_j)^*$ in its support. If Y factors in $C[M]^W$ then each factor is one such subproduct by the *UFD* property of $C[\Lambda]^W$. Therefore, a factoring provides b_j for $j = 1, \dots, n$ such that $0 \leq b_j \leq a_j$, not all $b_j = a_j$, and both $\sum b_j w_j$ and $\sum (a_j - b_j) w_j$ lie in M . This contradicts the M -indecomposability of $\sum a_j w_j$. In summary, Y is an irreducible element in $C[M]^W$.

Let d be the index of M in Λ . Then $d w_j \in M$ for each fundamental dominant weight w_j . Again, Lemma 15 yields

$$X(w_j)^d \in C[M]^W \quad \text{for } j = 1, \dots, n.$$

Consider the equation

$$Y^d = [X(w_1)^d]^{a_1} [X(w_2)^d]^{a_2} \cdots [X(w_n)^d]^{a_n}$$

inside $C[M]^W$. Since Y is irreducible, $Y \mid X(w_k)^d$ for some k . Interpret this in $C[\Lambda]^W$ and use unique factorization there: $Y = X(w_k)^{a_k}$. That is, the M -indecomposable weights all have the form $a_k w_k$.

If $a_k w_k$ and $a'_k w_k$ lie in M , so does $\text{GCD}(a_k, a'_k) w_k$. But $\text{GCD}(a_k, a'_k)$ divides both a_k and a'_k . By indecomposability, there are no such repeats:

$$r_1 w_1, \dots, r_n w_n \quad (r_j > 0 \text{ an integer})$$

is a complete list of the M -indecomposable elements. (Notice that some positive integer multiple of each w_j must be M -indecomposable.) They are clearly linearly independent over \mathbf{Z} . The argument is completed by showing that they span M . Suppose $\sum_{i=1}^n c_i w_i \in M$. Choose a large positive integer N

such that $\frac{c_i}{r_i} \leq N$ for $i = 1, \dots, n$. Since $r_i w_i \in M$ we have $N(\sum_{i=1}^n r_i w_i) \in M$.

Thus $\sum_{i=1}^n (Nr_i - c_i) w_i \in M$. Since $Nr_i - c_i \geq 0$,

$$\sum_{i=1}^n (Nr_i - c_i) w_i \in M \cap \Lambda^+.$$

Now every member of $M \cap \Lambda^*$ is a sum of M -indecomposable elements. Solve for $\sum c_i w_i$. □

Finally, we can put together Theorem 8, Proposition 12, and Theorem 16. We cite the fact that a reflection group may appear as the Weyl group for more than one root system. By replacing certain component root systems of type B_n with those of type C_n , every stretched weight lattice over a given reflection group becomes isomorphic, as an abstract module, to some ordinary weight lattice. (See § 1 and the "note added in proof" of [5].)

MAIN THEOREM. Assume A is a \mathbb{Z} -lattice and $G \subset GL(A)$ is a finite group. Then $\mathbb{C}[A]^G$ is a polynomial ring if and only if G is a reflection group and, for some choice of root system, it becomes a Weyl group with A as its weight lattice.

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NOTE ADDED IN PROOF: As occasionally happens when a mathematician wanders from his area of expertise, he re-invents the wheel. The appendix (§ 4) can be eliminated by invoking a theorem of Serre [B] to the effect that the fixed ring of a suitably nice regular local ring under the action of a finite group is also regular local if and only if the group acts as a pseudo-reflection group on the tangent space of the original local ring. The fifth section is, to a large extent, implicit in work of Steinberg [C]. A statement closer to mine can be found in [A].

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