2. NOTATIONS. In what follows, lower case latin letters stand for rational integers. In particular, p, with or without subscripts, stands for primes $p \equiv 1 \pmod 4$ and q for primes $q \equiv 3 \pmod 4$. Whenever the modulus is not specified, it will be understood that the congruence is taken modulo 4. Also, n_1 stands for a positive integer divisible only by primes $p \equiv 1$ (which does not exclude $n_1 = 1$) and n_2 for a positive integer divisible only by primes $q \equiv 3$ ($n_2 = 1$ is possible). The symbol $[x]$ stands for the greatest integer not in excess of x and $a^b \| c$ means that $a^b \mid c$, but $a^{b+1} \nmid c$. If $f(x)$ is asymptotically equal to $g(x)$, i.e., if $f(x)/g(x) \to 1$ for $x \to \infty$, we write $f(x) \simeq g(x)$. A weaker relation will be defined in Section 3.

3. MAIN RESULTS. The main results are formulated in the following two theorems.

THEOREM 1.

(a) *For all* $k \geqslant 5$, $S_{k,0} = \emptyset$; *for* $m \geqslant 1$, *the sets* $S_{k,m}$ *are finite and can be determined explicitly.*

(b) *For* $k = 4$, $S_{4,0} = \emptyset$; *for* $m \geqslant 1$, $S_{4,m} = S^{(1)}_{4,m} \cup S^{(2)}_{4,m}$, *with* $S^{(1)}_{4,m}$ *a finite, computable set and* $S^{(2)}_{4,m} = \{n \mid n = 4^a t, t \in N_m\}$, *where* N_m *are finite, computable sets.*

(c) *For* $k = 3$, $S_{3,0} = \{n \mid n = 4^a t, t \equiv 7 \pmod 8\}$, *while, for* $m \geqslant 1$, $S_{3,m} = \{n \mid n = 4^a t, t \in M_m\}$, *with* M_m *finite sets.*

(d) *For* $k = 2$, $S_{2,0} = \{n \mid \exists\, q \text{ prime}, q \equiv 3, q^{2u+1} \| n\}$; *for* $m \geqslant 1$, $S_{2,m} = \{n = 2^a n_1 n_2^2\}$, *where the factors* $p_i^{b_i}$ *of* n_1 *have exponents* b_i *that are solutions of the equation*

$$\left[\frac{1}{2}\Big(1 + \prod_{p_i^{b_i} \| n} (b_i + 1)\Big)\right] = m. \tag{1}$$

(e) *For* $k = 1$, $S_{1,0} = \{n \neq t^2\}$, $S_{1,1} = \{n \mid n = t^2\}$ *and, for* $m \geqslant 2$, $S_{1,m} = \emptyset$.

All sets $S_{k,m}$ can be effectively computed, except for $S_{3,m}$. The determination of the $S_{3,m}$ depends on the complete listing of the discriminants $d = -n$ of binary quadratic forms of class numbers $h(-n) \leqslant 2m$ for $n \equiv 3 \pmod 8$ and $h(-4n) \leqslant 4m$ for $n \equiv 1, 2 \pmod 4$, respectively.

It also is of some interest to determine the sets $S_{k,m}(x)$ and the numbers $|S_{k,m}(x)|$ of their elements, $n \leqslant x$, $n \in S_{k,m}$. In order to avoid exceptions that are basically trivial, but would lead to longwinded formulations, in the statements of the following Theorem 2, we shall assume that x is "large", i.e., that x is larger than the largest element of any finite set which occurs in any given formula.

For a neat formulation of Theorem 2, it also is convenient to define the following sets. For given m, let $0 < b_1 \leqslant b_2 \leqslant \ldots \leqslant b_r$ be a set of integers that satisfies (1). Then we denote by $S_{2,m}^{(b_1, b_2, \ldots, b_r)}$ the set of integers $n = 2^a n_2^2 p_1^{b_1} p_2^{b_2} \ldots p_r^{b_r}$; here $p_1, p_2, \ldots, p_r$ run independently through all primes $p \equiv 1$, with the restriction that, for $i \neq j$, $p_i \neq p_j$ and (to avoid multiplicities) $p_i < p_j$ if $i < j$ and $b_i = b_j$. The set of integers that satisfy (1) and of the form $n = 2^a n_2^2$ is denoted by $S_{2,m}^{(\varnothing)}$; it is clear that $S_{2,m}^{(\varnothing)} \neq \varnothing$ if and only if $m = 1$. The set $S_{2,m}$ of all integers that have exactly m partitions into 2 squares is the union $\cup S_{2,m}^{(b_1, b_2, \ldots, b_r)}$ taken over all solution sets of (1). The meaning of a notation like $S_{2,m}^{(b_1, b_2, \ldots, b_r)}(x)$ or $| S_{2,m}^{(b_1, \ldots, b_r)}(x) |$ is clear.

We shall say that $f(x)$ is *weakly asymptotic* to $g(x)$, in symbols $f(x) \doteq g(x)$, if, to every $\varepsilon > 0$, there exists $x_0 = x_0(\varepsilon)$, such that, for $x \geqslant x_0$, it is not possible that either of the two inequalities, $f(y) > (1+\varepsilon)g(y)$, or $f(y) < (1-\varepsilon)g(y)$, should hold over the whole interval $x \leqslant y \leqslant x^{1+\varepsilon}$. The relation is indeed weaker than ordinary asymptotic equality, in that it does not rule out that one of said inequalities should hold somewhere within the stated interval.

THEOREM 2.

(a) *For* $k \geqslant 5$, *all sets* $S_{k,0}$ *are empty, while* $| S_{k,m}(x) |$ *are finite, computable integers for all* $m \geqslant 1$.

(b) *For* $k = 4$, *the set* $S_{4,0}$ *is empty and, if* $m \geqslant 1$,

$$| S_{4,m}(x) | = | S_{4,m}^{(1)} | + (\log 4)^{-1}(| N_m | \log x - \sum_{t \in N_m} \log t) + \varepsilon_{m,x};$$

here N_m *are the same sets as in Theorem 1 and* $0 \leqslant \varepsilon_{m,x} \leqslant | N_m |$.

(c) *For* $k = 3$, *if* $m = 0$, *then* $| S_{3,0}(x) | = \dfrac{x}{6} + O(\log x)$; *if* $m \geqslant 1$, *then*

(2) $$| S_{3,m}(x) | = (\log 4)^{-1}(| M_m | \log x - \sum_{t \in M_m} \log t) + \delta_{m,x},$$

where M_m *are the same sets as in Theorem 1 and* $0 < \delta_{m,x} \leqslant | M_m |$.

(d) *For* $k = 2$, *if* $m = 0$, *then*

$$| S_{2,0}(x) | = x - bx(\log x)^{-1/2}$$
$$+ o(x(\log x)^{-1/2}),\ b = \{2 \prod_{q \equiv 3} (1-q^{-2})\}^{-1/2}.$$

If $m \geqslant 1$, *then*

(3a) $$| S_{2,1}^{(\varnothing)}(x) | \simeq b(x/\log x)^{1/2} ;$$

for $(b_1, b_2, \dots b_r) \neq (\varnothing)$,

(3b) $$| S_{2,m}^{(b_1, b_2, \dots, b_r)}(x) | \simeq C(x/\log x)^{1/2}, \quad \text{if all} \quad b_i \geqslant 3 ;$$

(3c) $$| S_{2,m}^{(b_1, b_2, \dots, b_r)}(x) |$$
$$\doteq \begin{cases} C(x/\log x)^{1/2}(\log_2 x)^k & \text{if} \quad 2 = b_1 = b_2 = \dots = b_k < b_{k+1} \leqslant \dots \leqslant b_r , \\ C(x/\log x)\,(\log_2 x)^{k-1} & \text{if} \quad 1 = b_1 = b_2 = \dots = b_k < b_{k+1} \leqslant \dots \leqslant b_r . \end{cases}$$

Here $\log_2 x = \log(\log x)$ *and we shall denote more generally* $\log(\log_{n-1} x)$ *by* $\log_n x$. *In (3a),* b *is an absolute constant, while in (3b) and (3c) the constant* C *depends on the corresponding set of the* $b_j, j = 1, 2, \dots, r$.

Finally,

$$| S_{2,m}(x) | = \sum | S_{2,m}^{(b_1, \dots, b_r)}(x) |$$

and

$$| S_{2,m}(x) | \doteq C_m \frac{x}{\log x} (\log_2 x)^{c(m)} ,$$

with C_m *and* $c(m)$ *computable constants,* $0 \leqslant c(m) \in \mathbf{Z}$.

(e) *For* $k = 1$, $| S_{1,0}(x) | = [x] - [\sqrt{x}]$, $| S_{1,1}(x) | = [\sqrt{x}]$ *and, for* $m > 1$, $| S_{1,m}(x) | = 0$.

Conjecture. All weak asymptotic equalities are actually ordinary asymptotic equalities.

The proof of this conjecture depends on a certain (apparently unproven) Tauberian statement.

Comments. Not only the finite sets $S_{k,m}$, $k \geqslant 5$, but also the infinite sets $S_{4,m}$ have density zero, i.e., $\lim_{x \to \infty} x^{-1} | S_{4,m}(x) | = 0$, notwithstanding the obvious fact that, if we set $S_k = \bigcup_{m=0}^{\infty} S_{k,m}$, then $S_k(x) = x$ and S_k has density 1 for all k.

Also $S_{k,m}$ with $k = 1, 2$, and 3 and $m \geqslant 1$ have density zero. On the other hand, $S_{3,0}$ has density 1/6, while $S_{2,0}$ and $S_{1,0}$ have density 1.

The cases $k = 1$, $k = 2$, and $k \geqslant 5$ hardly require further comments and follow trivially from Theorem 1 and 2. On the other hand, the situation is not quite so obvious for $k = 3$ and $k = 4$. By Theorem 2(c),

$$\sum_{m=1}^{\infty} S_{3,m}(x) = \frac{5x}{6} + O(\log x), \tag{4}$$

a well-known result (see, e.g., [7], vol. 2, § 176, p. 645), but which does not seem to follow from (2). Indeed, $S_{3,m} \cap S_{3,m'} = \emptyset$ for $m \neq m'$ and this implies that, if we set $S'_3 = \bigcup_{m=1}^{\infty} S_{3,m}$, then $|S'_3(x)| = |\bigcup_{m=1}^{\infty} S_{3,m}(x)|$ $= \sum_{m=1}^{\infty} |S_{3,m}(x)|$. If we replace here $|S_{3,m}(x)|$ by (2), the last series diverges. The reason for this fact is that, for any fixed x, the summands are given by (2) only for the finitely many values of m with $\max(j \mid j \in M_m) \leqslant x$. The other summands have to be replaced by smaller values and there is no contradiction with (2). For $k = 4$ the situation is similar.

4. Proof of Theorem 1. It is known (see [1], especially p. 71, and [12]) that, for $k \geqslant 3$, $r_k(n) = c_k(n)n^{k/2-1}(1+o(1))$. For $k \geqslant 5$, $c_k(n) \geqslant C_k > 0$, inequalities that do not hold for $k = 3$, and $k = 4$. Hence, for $k \geqslant 5$,

$$P_k(n) \geqslant r_k(n)/2^k k! \geqslant C_k n^{k/2-1}/2^k k! > m$$

if

$$n^{k/2-1} > (2^k k!/C_k)m, \quad \text{or} \quad n > n_0(k, m) = \left(\frac{2^k k!}{C_k} m\right)^{2/(k-2)}$$

It is sufficient to check the finitely many $n \leqslant n_0$ and keep those for which $P_k(n) = m$. Incidentally, in this search, we automatically obtain also the sets $S_{k,m'}$ for all $m' < m$. This proves (a).

For $k = 4$, by Lagrange's Theorem, $5_{4,0} = \emptyset$. We now consider $m \geqslant 1$. We recall that $P_4(8n) = P_4(2n)$. Also (see [8]), for $8 \nmid n$, $P_4(n) > n/48$, for $n \not\equiv 0 (\text{mod } 4)$, $P_4(n) > n/64$ for $n \equiv 4 (\text{mod } 8)$. In either case, $P_4(n) > m$, for all $n \not\equiv 0 (\text{mod } 8)$, $n > 64m$. Consequently, if $n \not\equiv 0 (\text{mod } 8)$, $P_4(n) = m$ is possible only for $n \leqslant 64m$. We now consider this set $\{n \leqslant 64m\}$ and eliminate from it the integers divisible by 8; let

$$S^{(3)}_{4,m} = \{n \leqslant 64m, n \not\equiv 0(\text{mod } 8) \mid P_4(n) = m\}.$$

Next, if $n \equiv 0(\text{mod } 8)$, we set $n = 4^a \cdot 2t$, with $2t \not\equiv 0(\text{mod } 8)$ (here t may be either even or odd). It follows that $P_4(n) = m$ implies $P_4(2t) = m, 8 \nmid 2t$, so that $2t \in S^{(3)}_{4,m}$ and $2t \leqslant 64m$. Consequently, for each $2t \in S^{(3)}_{4,m}$, all integers $n = 4^a \cdot 2t$ have $P_4(n) = m$; we denote the set of these integers by $S^{(4)}_{4,m}$. Next we consider the set of odd elements $j \in S^{(3)}_{4,m}$; for all of them, $P_4(j) = m$. For each of them we check whether $P_4(4j) > m$, or $P_4(4j) = m$. In the first case, $P_4(4^a j) > m$ for all $a \geqslant 1$; in the second case (which in fact does not occur), $P_4(4^a j) = m$ for all $a \geqslant 0$. The union of the latter sets and $S^{(4)}_{4,m}$ is denoted by $S^{(2)}_{4,m}$ (in fact, $S^{(2)}_{4,m} = S^{(4)}_{4,m}$, but that is not important and we suppress the proof). The remaining elements $j \in S^{(3)}_{4,m}$, all odd, for which $P_4(4j) > m$, form the finite set $S^{(1)}_{4,m}$ and this finishes the proof of Theorem 1(b).

For $k = 3$, it is classical (see, e.g., [7], vol. 2 § 176, p. 644) that $S_{3,0} = \{n \mid n = 4^a(8j+7)\}$. It also is clear that $P_3(4n) = P_3(n)$. It is, therefore, sufficient to determine only the set M_m of integers $t \not\equiv 0(\text{mod } 4)$, with $P_3(t) = m$, as then $S_{3,m} = \{n \mid n = 4^a t, 4 \nmid t, t \in M_m\}$.

As already observed, $P_3(n) \geqslant r_3(n)/2^3 3! = r_3(n)/48 \geqslant R_3(n)/48$, where $R_3(n)$ stands for the number of *primitive* representations of n as a sum of 3 squares. Here primitive means that the 3 summands have no non-trivial common divisor. It is well-known (see [4], § 281, or [1]) that $R_3(n) = 24h(-n)$, if $n \equiv 3(\text{mod } 8)$, $n \neq 3$ and $R_3(n) = 12h(-4n)$, if $n \equiv 1, 2(\text{mod } 4)$, $n \neq 1$. Here $h(-n)$, or $h(-4n)$ is the class number of the quadratic forms of discriminant $d = -n$, or $d = -4n$, respectively. It follows that, if $h(-n) > 2m$, or $h(-4n) > 4m$, respectively, then $P_3(n) > m$. It is known that, for $d < 0$, $h(d)$ can take any given value only a finite number of times. Hence, for any m, there exists a finite integer $n_0 = n_0(m)$, such that, if $n > n_0$, then $h(-n)$ exceeds $2m$, or $h(-4n)$ exceeds $4m$, respectively. Consequently, it is sufficient to examine the finitely many integers $1 \leqslant n \leqslant n_0$ and keep among them the set $M_m = \{t \leqslant n_0, t \not\equiv 0(\text{mod } 4), P_3(t) = m\}$. Then $P_3(n) = m$ will hold if, and only if $n = 4^a t$, with $t \in M_m$. This essentially completes the proof of Theorem 1(c).

Unfortunately, we are not able to compute effectively $n_0(m)$ for arbitrary m, because we only know the complete list of discriminants $d < 0$ for which $h(d) = 1$, or $h(d) = 2$ (see [10]). This is sufficient for the determination of the integers $n \equiv 3(\text{mod } 8)$, for which $P_3(n) = 1$; but even the determination of the integers $n \equiv 1, 2(\text{mod } 4)$ with $P_3(n) = 1$ requires the knowledge of the discriminants $d < 0$, with $h(d) = 4$ (for combinatorial reasons, knowledge of the discriminants $d < 0$, with $h(d) = 3$ is not needed; see [2] for details).

Buell [3] has determined computationally all discriminants $-d \leqslant 10^6$ that correspond to class numbers $h(d) \leqslant 125$. Most likely, there are no other discriminants with $h(d) \leqslant 125$ beyond $-d = 10^6$; however, there exists no proof of this conjecture. If we assume that Buell's list is complete, then we can determine effectively the corresponding values of n, for $n \equiv 3 \pmod 8$ up to $m = 60$ and for $n \equiv 1, 2, 5, 6 \pmod 8$, up to $m = 30$. Out of these finite sets of integers $n \leqslant n_0(m)$, it is, of course, a simple matter to select those that lead to $P_3(n) = m$. This completes the proof of Theorem 1(c).

For $k = 2$, it is classical that, if we set $n = 2^a n_1 n'$, where n_1 contains only p's and n' only q's, then n has no representation as a sum of two squares unless $n' = n_2^2$. If this condition holds, and $n_1 = \prod_{p_i \equiv 1} p_i^{b_i}$, then $r_2(n) = 4\prod(b_i+1)$. If $n \neq b^2$, or $2b^2$, then both summands are different from zero and distinct, so that $P_2(n) = r_2(n)/2^2 2! = \frac{1}{8} r_2(n) = \frac{1}{2}\prod(b_i+1)$. The conditions imposed insure that at least one of the b_i is odd, so that $\frac{1}{2}\prod(b_i+1)$ is indeed an integer. In the excluded cases, the exceptional partitions $n = b^2 + 0$, or $n = b^2 + b^2$ correspond each to only 4 representations; hence,

$$8(P_2(n)-1) + 4 = r_2(n) \text{ and } P_2(n) = \frac{1}{8}(r_2(n)+4) = \frac{1}{2}\{\prod(b_i+1)+1\},$$

with $\prod(b_i+1)$ odd. It follows that, if $P_2(n) = m$, then the exponent b_i have to be solutions of $\prod(b_i+1) = 2m - \delta$, $\delta = 1$, if $n = b^2$, or $2b^2$, $\delta = 0$ otherwise. In either case, the b_i are a solution set of (1). The number of these solution sets with $0 \leqslant b_i \in \mathbf{Z}$ is obviously finite for any given m. The result obtained is equivalent to the statement (d) of Theorem 1.

The case $k = 1$ is trivial and this finishes the proof of Theorem 1.

5. Proof of Theorem 2. Statements (a), (b), (e) and (2) follow immediately from Theorem 1. The statements concerning $|S_{3,0}(x)|$ and $|S_{2,0}(x)|$ are due to Landau (see [7], § 176-183) and have been given here only for completeness. It only remains to prove the other statements of (d).

6. AN AUXILIARY THEOREM. The proof of Theorem 2(d) is unexpectedly difficult. We start by recalling the concepts of slowly varying functions and of functions of regular growth.

Definition 1. A function $L(x)$ is said to be *slowly varying* if, for every $r > 0$, $\lim_{x\to\infty} (L(rx)/L(x)) = 1$.

Definition 2. A function $\phi(x)$ is said to be of *regular growth* if

$$\lim_{x\to\infty} \frac{1}{x\phi(x)} \int_0^x \phi(t)dt = \frac{1}{\gamma+1}, \quad \text{with } \gamma \geqslant 0 .$$

For later use, we remark that all the functions of the type $(\log_n x)^a$, as well as their quotients and their products are slowly varying. Also, functions of the type $\phi(x) = x^\gamma L(x)$ with $\gamma > 0$ and $L(x)$ slowly varying are of regular growth.

THEOREM A. (Karamata, see [6]). *Let* $A(x)$ *be a nondecreasing function and* $\phi(x) = x^\gamma L(x)$, *with* $L(x)$ *slowly varying and* $\gamma > 0$, *so that* $\phi(x)$ *is of regular growth, with* $\phi(x) \to \infty$ *for* $x \to \infty$. *Assume, furthermore, that* $f(s)$ *is defined by the Stieltjes integral*

$$f(s) = \int_0^\infty e^{-st} d\{A(t)\} ,$$

convergent for $s > 0$. *Then, if* $f(s) \simeq \phi(s^{-1})$ *for* $s \to 0$, *it follows that, as* $x \to \infty$,

$$A(x) \simeq \phi(x)/\Gamma(\gamma+1) .$$

We shall use Theorem A only in the particular case in which $A(x) = \sum_{j=1}^{[e^x]} a_j$, where all $a_j \geqslant 0$. In that case, the Stieltjes integral that defines $f(s)$ becomes a Dirichlet series, $\int_0^\infty e^{-st} d\{A(t)\} = \sum_{n=1}^\infty a_n e^{-s\log n} = \sum_{n=1}^\infty \frac{a_n}{n^s}$.

7. PROOF OF THEOREM 2(d). Consider the set $S_{2,1}^{(\varnothing)} = \{n | n = 2^a n_2^2\}$. The generating function of the integers $n \in S_{2,1}^{(\varnothing)}$ is

$$f_0(s) = \frac{1}{1-2^{-s}} \prod_{q\equiv 3} \frac{1}{1-q^{-2s}} = \sum_{n=1}^\infty \frac{a_n}{n^s},$$

where $a_n = 1$ if $n \in S_{2.1}^{(\emptyset)}$, $a_n = 0$ otherwise.

By Perron's Theorem (see, e.g., [11], p. 376):

$$S(x) \underset{\text{def}}{=} \sum_{n \leqslant x}' a_n = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} f_0(s) \frac{x^s}{s} \, ds .$$

Here $\sum_{n \leqslant x}'$ means that for $x \in \mathbf{Z}$ the term a_x is replaced by $\frac{1}{2} a_x$; however, in order to avoid this trivial, but cumbersome situation, henceforth we shall assume, without loss of generality, that $x \notin \mathbf{Z}$ and suppress the prime.

The integral is taken along a parallel to the imaginary axis, of abscissa $\sigma > 0$, sufficiently large to insure convergence, say $\sigma(=\operatorname{Re} s) = 1$. In fact, the first singularity of the integral occurs at $s = 1/2$ and it is a branch point. In order to compute the integral, it is convenient to change variables, so as to bring the singularity to $s = 1$. For that reason, we set $f(s) = f_0(s/2)$ and obtain

$$S(s) = \frac{1}{2\pi i} \int_{2\sigma - i\infty}^{2\sigma + i\infty} f(s) \frac{x^{s/2}}{(s/2)} \, d(s/2) = \frac{1}{2\pi i} \int_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} f(s) \frac{y^s}{s} \, ds ,$$

where $y = \sqrt{x}$ and $\sigma_1 = 2\sigma$. The series $\sum_{n=1}^{\infty} a_n n^{-s/2}$ that represents $f(s)$ in $\sigma > 1$ is, of course, no longer a Dirichlet series, but that is irrelevant to the computation of the integral, for which we use the method of Landau (see [7], §§ 180-183 for more details). We consider the contour ABCDEFGHA. Along the arcs BC and FG, with $|t| > 3$, we have $\sigma = 1 - c/(\log|t|)$ and, for $|t| \leqslant 3$ the abscissa from C to F stays constant, $\sigma = 1 - c/(\log 3) = \tau$, say; here the constant c is selected in such a way that both $\zeta(s)$ and $L(s) = L(s, \chi)$ $(\chi(n) =$ non-principal character modulo 4) have no zeros inside and on the contour. We now cut the plane from τ to 1 and replace the segment DE by the path DL on the upper rim of the cut, followed by an arc of circle around $s = 1$, of "small" radius and the segment KE, along the lower rim of the cut. In the cut plane we have, inside and on the contour,

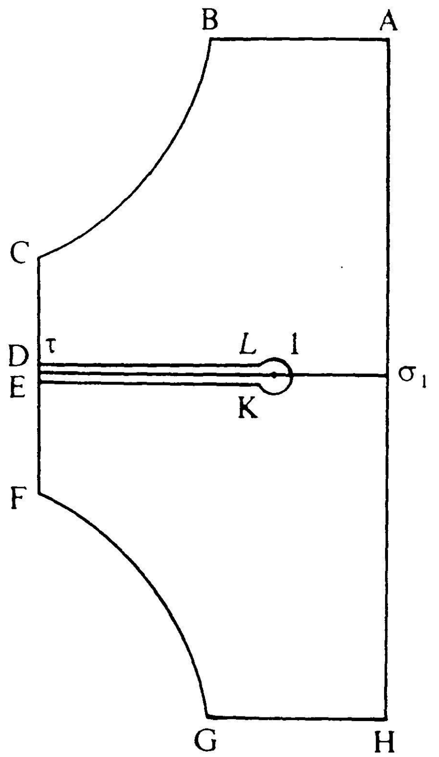

FIGURE 1

$$\log f(s) = -\log\left(1 - \frac{1}{2^{s/2}}\right) - \sum_{q\equiv 3} \log\left(1 - \frac{1}{q^s}\right) = \sum_{q\equiv 3} \frac{1}{q^s} + g_1(s),$$

defined as a singlevalued function; here $g_1(s)$, as well as all other functions $g_j(s)$ that will occur, are holomorphic for $\sigma > 1/2$. Also,

$$\frac{1}{2}\{\log \zeta(s) - \log L(s)\} = \sum_{q\equiv 3} \frac{1}{q^s} + g_2(s).$$

It follows that

$$\log f(s) = \frac{1}{2}\{\log \zeta(s) - \log L(s)\} + g_3(s)$$

and that

$$\log \frac{f(s)}{s} + \frac{1}{2}\log(s-1) = \frac{1}{2}\{\log((s-1)\zeta(s)) - \log L(s)\} + g_4(s),$$

where now both sides are holomorphic inside and on the contour, *without* the cut. There, also $\log L(s)$ is holomorphic, so that $\frac{f(s)}{s} = \frac{1}{2}\log\frac{1}{s-1}$ $+ g(s)$, with $g(s)$ holomorphic inside and on the contour without the cut (but not necessarily in all of $\sigma > 1/2$). It follows that, when we turn around the point $s = 1$, the argument of $f(s)$ decreases by π and we conclude that

$$\text{(5)} \qquad \frac{f(s)}{s} = \frac{1}{\sqrt{(s-1)}}\left(A + \sum_{i=1}^{\infty} A_i(s-1)^i\right),$$

with $A \neq 0$ holds along the contour. We now compute the integral. It is wellknown (see, e.g., [7], or [11]) that the integral, computed along the arcs ABCD and EFGH (with either large values of $|s| \geqslant T$, or small values of y^σ, $\sigma < 1$), contributes only $O(ye^{-\alpha\sqrt{(\log y)}})$. Also $\lim_{T\to\infty} \int_{\sigma_1 - iT}^{\sigma_1 + iT}$ differs from the finite integral by a similar error term, so that

$$S(x) = \frac{1}{2\pi i}\int_{\text{EKLD}} + E(y) = -\frac{1}{2\pi i}\int_{\text{DLKE}} + E(y),\ E(y) = O(ye^{-\alpha\sqrt{(\log y)}}),$$

or

$$-2\pi i\, S(x) = \int_{\text{DLKE}} f(s)\frac{y^s}{s}\,ds + O(ye^{-\alpha\sqrt{(\log y)}}).$$

The main term of the integral is

$$A \cdot \int_{\mathrm{DLKE}} \frac{y^s}{\sqrt{(s-1)}}\, ds = A(e^{-i\pi/2} - e^{i\pi/2}) \int_\tau^1 \frac{y^s}{\sqrt{(1-s)}}\, ds$$

$$= -2iA\left\{\int_0^1 \frac{y^s}{\sqrt{(1-s)}}\, ds - E_1\right\};$$

here

$$|E_1| = \int_0^\tau \frac{y^s}{\sqrt{(s-1)}}\, ds < y^\tau \int_0^\tau \frac{d(s-1)}{\sqrt{(1-s)}} = O(y^{1-c/(\log 3)}) = o(ye^{-\alpha\sqrt{(\log y)}})$$

and will be absorbed into the error term. Also,

$$\int_0^1 \frac{y^s}{\sqrt{(1-s)}}\, ds = y\int_0^1 \frac{y^{s-1}}{\sqrt{(1-s)}}\, d(s-1) = -y\int_1^0 u^{-1/2}y^{-u}du$$

$$= y\int_0^1 e^{-u\log y}u^{-1/2}du = \frac{y}{\sqrt{(\log y)}}\int_0^{\log y} e^{-v}v^{-1/2}\, dv$$

and, for $y \to \infty$, the integral equals $\sqrt{\pi} + O(y^{-1}(\log y)^{-1/2})$. The other terms of $f(s)$ form a convergent series and their contribution to the integral is easily found to be

$$O\left\{\int_{\mathrm{DLKE}} y^s\sqrt{(1-s)}ds\right\} = O\left\{\int_0^1 y^s\sqrt{(1-s)}ds\right\} = O\big(y(\log y)^{-3/2}\big)$$

$$= o\big(y(\log y)^{-1/2}\big).$$

Putting these results together, we obtain

$$S(x) = \sum_{n\leqslant x} a_n = -\frac{2iA}{2\pi i}\sqrt{\pi}\frac{y}{\sqrt{(\log y)}}\big(1+o(1)\big) = \frac{y}{\sqrt{(\log y)}}\left(\frac{A}{\sqrt{\pi}} + o(1)\right).$$

From (5) we find that $A = \lim_{s\to 1}\sqrt{(s-1)}f(s)$. From the definition of $f(s)$ we find

$$A^2 = \lim_{s\to 1}\left\{(s-1)(1-2^{-s/2})^{-2}\prod_{q\equiv 3}(1-q^{-s})^{-2}\right\}$$

$$= \lim_{s\to 1}\left\{\zeta(s)^{-1}(1-2^{-s/2})^{-2}\prod_{q\equiv 3}(1-q^{-s})^{-2}\right\}$$

$$= \lim_{s\to 1}\left\{\frac{1+2^{-s/2}}{1-2^{-s/2}}\prod_{p\equiv 1}(1-p^{-s})\prod_{q\equiv 3}(1-q^{-s})^{-1}\right\}$$

$$= \lim_{s\to 1}\left\{\frac{1+2^{-s/2}}{1-2^{-s/2}}\cdot\frac{1}{L(s)}\cdot\prod_{q\equiv 3}(1-q^{-2s})^{-1}\right\}$$

$$= (1+\sqrt{2})^2\cdot\frac{4}{\pi}\cdot\prod_{q\equiv 3}(1-q^{-2})^{-1}.$$

It follows that

$$S(x) = \sum_{n \leqslant x} a_n \simeq \frac{A}{\sqrt{\pi}} \frac{y}{\sqrt{(\log y)}} = \frac{2(\sqrt{2}+1)}{\pi} \frac{\sqrt{x}}{\sqrt{\left(\frac{1}{2}\log x\right)}} \left\{\prod_{q\equiv 3}\left(1-\frac{1}{q^2}\right)\right\}^{-1/2}$$

$$= \frac{2(2+\sqrt{2})}{\pi}\left\{\prod_{q\equiv 3}\left(1-\frac{1}{q^2}\right)\right\}^{-1/2} \sqrt{(x/(\log x))}\,.$$

This finishes the proof of (3a), with $b = \frac{2(2+\sqrt{2})}{\pi}\left\{\prod_{q\equiv 3}(1-q^{-2})\right\}^{-1/2}$ $\simeq 2.3490 \ldots$ (based on $\prod_{q\equiv 3}(1-q^{-2})^{-1} \simeq 1.1680$).

In order to gain some insight into the speed of convergence to this limiting value, we observe that

$$S(320{,}000) = \sum_n a_n \,(n \in S_{2,1}^{(\varnothing)},\, n \leqslant 320{,}000) = 372\,,$$

and

$$372 \Big/ \sqrt{\frac{320{,}000}{\log 320{,}000}} \simeq 2.3413$$

which is already fairly close to b.

8. Proof of Theorem 2(d) (continuation). Let us consider the set $S_{2,m}^{(b_1,\ldots,b_r)} = \{n | n = 2^a n_2^2 p_1^{b_1} p_2^{b_2} \ldots p_r^{b_r}\}$, with $3 \leqslant b_1 \leqslant b_2 \leqslant \ldots \leqslant b_r$. The generating function is

$$f_1(s) = \frac{1}{1-\frac{1}{2^s}} \prod_{q\equiv 3} \frac{1}{1-\frac{1}{q^{2s}}} \sum{}^{*} \prod_{i=1}^{r} p_i^{-b_i s} = \sum_{n=1}^{\infty} \frac{a_n}{n^s}\,,$$

with $a_n = 1$ if $n \in S_{2,m}^{(b_1,\ldots,b_r)}$, $a_n = 0$ otherwise.

Here the star means that we sum over products $p_1^{-b_1 s} p_2^{-b_2 s} \ldots p_r^{-b_r s}$, where the p_i run independently through the primes $p \equiv 1$, with the following restrictions: for $i \neq j$, also $p_i \neq p_j$ and furthermore, in order to avoid counting more than once a product like $p_i^{-bs} p_j^{-bs}$ with the same exponent, we require that if $b_i = b_j$ then for $i < j$, also $p_i < p_j$. When convenient we shall denote the function represented by this sum simply by $\Sigma^*(s)$. For future use we observe that $\sum^* \prod_{i=1}^{r} p_i^{-bs}$ can be written as a sum of products of series of the type $\sum_p p^{-bs}$ and we shall be interested mainly in the product with the largest abscissa of convergence. So, e.g., $\sum^* p_1^{-s} p_2^{-2s} = (\sum_p p^{-s})(\sum_p p^{-2s}) - \sum_p p^{-3s}$. The behaviour of this sum depends mainly on the first term, with abscissa of convergence $\sigma = 1$, while the last term converges for $\sigma > 1/3$. Similarly,

$$\sum^* p_1^{-s} p_2^{-s} = \sum_{p_1 < p_2} p_1^{-s} p_2^{-s} = \frac{1}{2}\{(\sum_p p^{-s})^2 - \sum_p p^{-2s}\}$$

$$= \frac{1}{2}(\sum_p p^{-s})^2 + g_1(s),$$

where the first term has the abscissa of convergence $\sigma = 1$, while $g_1(s)$ is holomorphic for $\sigma > 1 - a$, with $a > 0$. More generally,

$$\sum^* \prod_{i=1}^{k} p_i^{-s} = \frac{1}{k!}(\sum_p p^{-s})^k + g_2(s),$$

with $g_2(s)$ holomorphic for $\sigma > 1 - a$, $a > 0$.

In the present case of $f_1(s)$, $\sum^*$ converges for $\sigma > 1/3$, on account of $b_j \geqslant 3 (j=1, 2, ..., r)$; hence, $f_1(s)$ is represented by its generating function for $\sigma > 1/2$ and, just as in the case of $f_0(s)$, it has a branch point at $s = 1/2$. One may follow exactly the previous computation and obtains the result

$$S(x) = \sum_{n \leqslant x} a_n \simeq C\sqrt{(x/(\log x))}, \quad \text{with } C = b\sum^*(1/2)$$

an this is precisely (3b).

Let us assume now that $2 = b_1 = b_2 = ... = b_k < b_{k+1} \leqslant ... \leqslant b_r$. The generating function (essentially the same as for $f_1(s)$, with the new values for the b_i's) may now be written as

$$f_2(s) = \frac{1}{1 - \dfrac{1}{2^s}} \prod_{q \equiv 3} \frac{1}{1 - \dfrac{1}{q^{2s}}} \sum^* p_1^{-2s} p_2^{-2s} \dots p_k^{-2s} p_{k+1}^{-b_{k+1}s} \dots p_r^{-b_r s}$$

$$= \frac{1}{1 - \dfrac{1}{2^s}} \prod_{q \equiv 3} \frac{1}{1 - \dfrac{1}{q^{2s}}} \left(\sum^* p_{k+1}^{-b_{k+1}s} \dots p_r^{-b_r s} + g_3(s)\right)\left(\frac{1}{k!}\left(\sum_p p^{-2s}\right)^k + g_4(s)\right).$$

Here the sum $\sum^*$ is holomorphic at least for $\sigma > 1/4$ and $g_3(s)$ even for $\sigma > \frac{1}{4} - a$, for some $a > 0$; also, $f_2(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ with $a_n = 1$ if

$$n \in S_{2,m}^{(b_1,\ldots,b_r)}, \quad a_n = 0$$

otherwise. As before we set $S(x) = \sum_{j \leqslant x} a_j$. The function $f_2(s)$ has a branch point at $s = 1/2$ and is holomorphic for $\sigma > 1/2$. As before, $S(x)$ will depend only on the behaviour of $f_2(s)$ in the neighborhood of its branch point.

For $s \to 1/2$, $\sum^*(s) + g_3(s) \to \sum^*(1/2) + g_3(1/2) = D$, say. We now obtain from (5) that

$$\frac{f(2s)}{2s} = \frac{f_0(s)}{2s} = \frac{1}{\sqrt{(2s-1)}}\left(A + O(2s-1)\right),$$

so that $f_0(s) = (1-2^{-s})^{-1} \prod_{q\equiv 3} (1-q^{-2s})^{-1} \simeq \dfrac{A}{\sqrt{2}} \dfrac{1}{\sqrt{(s-1/2)}}$

and

$$f_2(s) \simeq \frac{A}{\sqrt{2}} \cdot D \cdot \frac{1}{\sqrt{(s-1/2)}} \left\{ \frac{1}{k!} (\sum_p p^{-2s})^k + g_4(s) \right\}.$$

To replace the last factor, we recall that

$$\log\zeta(2s) = \sum_{p\equiv 1} p^{-2s} + \sum_{q\equiv 3} q^{-2s} + h_5(s) \text{ and } \log L(2s)$$
$$= \sum_{p\equiv 1} p^{-2s} - \sum_{q\equiv 3} q^{-2s} + h_6(s),$$

where $h_5(s)$, $h_6(s)$, etc., are holomorphic for $\sigma > 1/4$.
It follows that

$$(6) \quad \sum_{p\equiv 1} p^{-2s} = \frac{1}{2}\left(\log\zeta(2s) + \log L(2s)\right) + h_7(s) = -\frac{1}{2}\log(2s-1) + h_8(s)$$
$$= -\frac{1}{2}\log\left(s - \frac{1}{2}\right) + h_9(s).$$

and finally

$$f_2(s) \simeq B\frac{1}{\sqrt{(s-1/2)}}\left(\left(\log\frac{1}{\sqrt{(s-1/2)}}\right)^k - \left(\log\frac{1}{\sqrt{(s-1/2)}}\right)^{k-2} g(1/2)\right)$$
$$\simeq \frac{B}{\sqrt{\left(s-\frac{1}{2}\right)}}\left(\log\left(s-\frac{1}{2}\right)^{-1/2}\right)^k, \quad \text{with } B = \frac{AD}{\sqrt{2}}.$$

In order to use Karamata's Theorem, we set $s - \frac{1}{2} = z$, so that

$$f_2(s) = \sum_{n=1}^{\infty} \frac{a_n/\sqrt{n}}{n^z} \simeq B\, z^{-1/2}(\log z^{-1/2})^k .$$

Let $\phi(z) = z^{1/2}(B \log^k z^{1/2})$; then $B(\log z^{1/2})^k$ is a slowly increasing function, $\phi(z)$ is of regular growth, with $\gamma = 1/2$ and, by Theorem A we obtain that

$$A(x) = \sum_{\log n \leqslant t = \log x} a_n/\sqrt{n} \simeq \frac{\phi(t)}{\Gamma\left(\frac{3}{2}\right)} = \frac{2}{\sqrt{\pi}} \cdot Bt^{1/2}(\log t^{1/2})^k = \frac{2^{1-k}}{\sqrt{\pi}} B\, t^{1/2}\log^k t$$

$$= \frac{2^{1-k}}{\sqrt{\pi}} B(\log x)^{1/2}(\log_2 x)^k .$$

We also have, with $S(x) = \sum_{n \leqslant x} a_n$, by partial summation, that

$$\sum_{n \leqslant x} \frac{a_n}{\sqrt{n}} = \sum_{n \leqslant x} \frac{S_n - S_{n-1}}{\sqrt{n}} = \sum_{n \leqslant x} S_n\left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{(n+1)}}\right) + \frac{S(x)}{\sqrt{x}}$$

$$= \frac{1}{2}\int_{a_1}^{x} \frac{S(y)}{y^{3/2}}\, dy + \frac{S(x)}{\sqrt{x}} .$$

The last term is at most $O(1)$, because the number of integers counted by $S(x)$ is of the same order as that of integers of the form $2^a n^2 \leqslant x$; hence,

(7) $$\sum_{n \leqslant x} \frac{a_n}{\sqrt{n}} \simeq \frac{1}{2}\int_{a_1}^{x} \frac{S(y)}{y^{3/2}}\, dy \simeq \frac{2^{1-k}}{\sqrt{\pi}} B(\log x)^{1/2}(\log_2 x)^k .$$

9. Some Tauberian considerations. The usual Tauberian theorems do not seem to apply (see, e.g., a very similar situation in [5], Theorem 3, where a crucial condition is $\alpha > 0$, while in the present case $\alpha = 0$).

If we assume that under the particular conditions of the present problem, it is legitimate to differentiate (7), we immediately obtain

(8) $$S(x) = |\, S_{2,m}^{(b_1, \dots, b_r)}(x)\,| \simeq C\sqrt{\left(\frac{x}{\log x}\right)}(\log_2 x)^k = F(x) ,$$

say, with $C = \frac{2^{1-k}}{\sqrt{\pi}} B$. Without that Tauberian conclusion, we can prove only that (8) holds in a certain weaker sense, defined in Section 3 as *weak asymptotic equality* and denoted by $\doteq$. The precise statement is the following

LEMMA. *Given* $\varepsilon > 0$, *arbitrarily small, there exists an* $x_0 = x_0(\varepsilon)$, *such that, for* $x \geqslant x_0$, *it is not possible that either* $S(y) \geqslant F(y)(1+\varepsilon)$, *or* $S(y) < F(y)(1-\varepsilon)$ *should hold over the whole interval* $x \leqslant y \leqslant x^{1+\varepsilon}$.

Proof. Assume first that $S(y) > F(y)(1+\varepsilon)$; then

$$\frac{1}{2}\int_x^{x^{1+\varepsilon}} \frac{S(y)}{y^{3/2}}\,dy > \frac{2^{-k}B}{\sqrt{\pi}}\,\varepsilon(1+\varepsilon)(1-\eta)(\log x)^{1/2}(\log_2 x)^k ,$$

where $\eta \to 0$ as $x \to \infty$. On the other hand, by (7),

$$\frac{1}{2}\int_x^{x^{1+\varepsilon}} \frac{S(y)}{y^{3/2}}\,dy < \frac{2^{1-k}B}{\sqrt{\pi}}(\log_2 x)^k(\log x)^{1/2}(\sqrt{(1+\varepsilon)}-1)$$

$$+\, o((\log x)^{1/2}(\log_2 x)^k) < \frac{2^{-k}B}{\sqrt{\pi}}\,\varepsilon(\log x)^{1/2}(\log_2 x)^k + o((\log x)^{1/2}(\log_2 x)^k)\,.$$

The asymptotic equality (7) now requires that $1 > (1+\varepsilon)(1-\eta)$, which is not the case for η sufficiently small (e.g., for $\eta < \varepsilon/2 < 1$), i.e., for $x \geqslant x_0(\varepsilon)$. The proof that $S(y) < F(y)(1-\varepsilon)$ cannot hold over the whole interval is similar.

For later use we observe that, if we replace B by its value, we obtain

$$C = \frac{2^{2-k}(1+2^{-1/2})}{\pi}\prod_{q\equiv 3}(1-q^{-2})^{-1}\cdot D\,.$$

This finishes the proof of the first result of (3c).

10. COMPLETION OF THE PROOF OF THEOREM 2. The last case to be considered is that of

$$S_{2,m}^{(b_1,\ldots,b_r)} = \{n \mid n = 2^a n_2^2 p_1 p_2 \ldots p_k p_{k+1}^{b_{k+1}} p_{k+2}^{b_{k+2}} \ldots p_r^{b_r}\}\,,$$

where $2 \leqslant b_{k+1} \leqslant \ldots \leqslant b_r$. The generating function of these integers is

$$f_3(s) = \frac{1}{1-\dfrac{1}{2^s}}\prod_{q\equiv 3}\frac{1}{1-\dfrac{1}{q^{2s}}}\sum{}^{*}\, p_1^{-s}\ldots p_k^{-s}p_{k+1}^{-b_{k+1}s}\ldots p_r^{-b_r s} = \sum_{n=1}^{\infty}\frac{a_n}{n^s},$$

with $a_n = 1$ if $n \in S_{2,m}^{(1,1,\ldots,1,b_{k+1},\ldots,b_r)}$, $a_n = 0$ otherwise. We observe that $f_3(s)$ may be written as

$$\frac{1}{1-2^{-s}}\prod_{q\equiv 3}\frac{1}{1-q^{-2s}}$$

$$\{\frac{1}{k!}(\sum_p p^{-s})^k(\sum{}^* p_{k+1}^{-b_{k+1}s}\dots p_r^{-b_rs}+g_5(s))+g_6(s)(\sum_p p^{-s})^{k-1}\},$$

with $g_6(s)$ holomorphic at least for $\sigma > 1/2$.

The function $f_3(s)$ is holomorphic for $\sigma > 1$, and has a branch point at $s = 1$. The behaviour of the summatory function $\sum_{n\leqslant x} a_n$ depends only on the behaviour of $f_3(s)$ in the vicinity of the branch point. There we have $\sum^*(1) + g_5(1) = P$, say, so that

$$f_3(s) \simeq \frac{2}{k!}\prod_{q\equiv 3}(1-q^{-2})^{-1}(\sum_{p\equiv 1} p^{-s})^k(\sum{}^*(s)+g_5(s))$$

and, as $s \to 1^+$,

$$f_3(s) \simeq \frac{2P}{k!}(\prod_{q\equiv 3}(1-q^{-2})^{-1})(\sum_{p\equiv 1} p^{-s})^k .$$

As before, we have

$$\sum_{p\equiv 1} p^{-s} = \frac{1}{2}(\log(\zeta(s)L(s))+h_9(s)) = -\frac{1}{2}\log(s-1) + h_{10}(s) .$$

It follows that, for $s \to 1^+$,

$$f_3(s) \simeq \frac{2P}{k!}\prod_{q\equiv 3}(1-q^{-2})^{-1}\left(\log\frac{1}{\sqrt{(s-1)}}\right)^k = Q(\log(s-1)^{-1/2})^k ,$$

say, with Q constant.

In order to apply Karamata's Theorem A, we set $s - 1 = z$ and rewrite $f_3(s)$ as $f_3(z+1) = \sum_{n=1}^{\infty}\frac{a_n/n}{n^z} \simeq Q(\log z^{-1/2})^k$. By Theorem A we obtain that

$$A(t) = \sum_{\log n\leqslant t = \log x}\frac{a_n}{n} \simeq Q(\log t^{1/2})^k \simeq 2^{-k}Q(\log_2 x)^k .$$

Proceeding as before, we set $S(x) = \sum_{n\leqslant x} a_n$ and, by partial summation, obtain

$$A(\log x) = \sum_{n\leqslant x}\frac{S_n - S_{n-1}}{n} = \sum_{n\leqslant x} S_n\left(\frac{1}{n}-\frac{1}{n+1}\right) + \frac{S(x)}{x+1} \simeq \frac{Q}{2^k}(\log_2 x)^k .$$

It is trivial to verify that $S(x) = O(x)$, so that we may neglect the term $S(x)/(x+1)$. Also,

$$\sum_{n\leqslant x} S_n\left(\frac{1}{n}-\frac{1}{n+1}\right) = \sum_{n\leqslant x} S_n \int_n^{n+1} \frac{du}{u^2} = \int_{a_1}^{x+1} \frac{S(u)}{u^2}\, du \simeq \int_{a_1}^{x} \frac{S(u)}{u^2}\, du .$$

By proceeding now exactly as in Section 9, we complete the proof of the last claim of (3c), with $C = \frac{2^{1-k}}{(k-1)!} P \prod_{q\equiv 3} (1-q^{-2})^{-1}$, with asymptotic equality at least in the weak sense. If the Tauberian Conjecture holds (i.e., if it is legitimate to differentiate our asymptotic equality), then the weak asymptotic equality is, in fact, an ordinary asymptotic equality.

Regardless of m, there always exist solutions with $a_1 = 1$. Indeed, $\prod(b_i+1) = 2m$ has at least the solution $b_1 = 1$, $b_2 = m-1$. This solution is unique only if m is a prime. More precisely, if $m = 2^c m_1$, $2 \nmid m_1$, then $2m = 2^{c+1}m_1 = \prod(b_i+1)$ admits the solution sets $1 = b_1 = b_2 = \ldots = b_{c+1} < b_{c+2} \leqslant \ldots \leqslant b_r$, where $b_{c+2}, \ldots, b_r$ are a solution set of

(9) $$\prod(1+b_i) = m_1 .$$

Each of these solution sets leads to an

$$S_{2,m}^{(1,\ldots,1,b_{c+2},\ldots,b_r)}(x) \simeq C \frac{x}{\log x} (\log_2 x)^{c+1} ,$$

with

$$C = C(b_{c+2}, \ldots, b_r; c, r) \quad \text{and} \quad S_{2,m} = \cup S_{2,m}^{(1,\ldots,1,b_{c+2},\ldots,b_r)} ,$$

and with the union taken over all solution sets of (9). It follows that $| S_{2,m}(x) | \doteq C_m \frac{x}{\log x} (\log_2 x)^{c+1}$, with $C_m = \Sigma\, C(b_{c+2}, \ldots, b_r; c, r)$ and this finishes the proof of Theorem 2.

11. EXAMPLES. To illustrate previous results numerically, we present here examples for all non-trivial cases (a) to (e).

(a) As an example for $k \geqslant 5$, let us study $S_{6,2}$ and $| S_{6,2}(x) |$, for $x \geqslant 15$ and for $x = 12$.

By evaluating, e.g., (91.5) in [12], it is easy to show that $r_6(n) > 4n^2$. Hence, $P_6(n) \geqslant r_6(n)/2^6 6! \geqslant (4/64.6!)n^2 = n^2/16.720$ and, for $n^2 > 2.16.720$, $P_6(n) > 2$. This means that $P_6(n) = 2$ is possible only for $n \leqslant 4^2.3.\sqrt{10} = 151.789\ldots$ It is well-known (see [8]) that $P_6(n) = 1$ holds precisely when $n = 1, 2, 3$, or 7. It remains to check $P_6(n)$ for the integers $n = 4, 5, 6$ and from 8 to 151. This is very easy with the help of a desk computer, but is not really difficult even with paper and pencil, especially by use of Gino Loria's tables [9] of the partitions of all integers up to 100 into (among others) four squares (attention is however due to some errors of those tables). The result is that $P_6(n) = 2$ precisely for

$$n \in S_{6,2} = \{4, 5, 6, 8, 10, 11, 15\}.$$

Clearly, for $x \geqslant 15$, $S_{6,2}(x) = S_{6,2}$ and $|S_{6,2}(x)| = 7$. On the other hand, $15 \notin S_{6,2}(12)$ and $|S_{6,2}(12)| = 6$.

(b) For $k = 4$, let us determine $S_{4,2}$ and compute $|S_{4,2}(x)|$ for x sufficiently large and also for $x = 100$ and $x = 20$.

By [8], if $n \not\equiv 0 \pmod 8$, then $P_4(n) > n/48$ for $n \not\equiv 0 \pmod 4$ and $P_4(n) > n/64$ for $n \equiv 0 \pmod 4$. Hence, if $n \geqslant 96$, $4 \nmid n$, then $P_4(n) > 2$. Similarly, if $n \geqslant 128$ and $n \equiv 4 \pmod 8$, then $P_4(n) > 2$. It is easy (either with the help of a computer, or manually by use of [9]) to find the values of all $P_4(n)$ for $n \leqslant 128$ and to record the integers n, with $P_4(n) = 2$. We obtain the set $\{4, 9, 10, 12, 13, 16, 17, 19, 20, 21, 22, 29, 30, 31, 35, 39, 40, 44, 46, 47, 48, 64, 71, 80, 88, 120\}$. By following the outlines of Section 4, we eliminate the integers divisible by 8 and so obtain the set

$$S^{(3)}_{4,2} = \{4, 9, 10, 12, 13, 17, 19, 20, 21, 22, 29, 30, 31, 35, 39, 44, 46, 47, 71\}.$$

The subset $N \subset S^{(3)}_{4,2}$ of even integers $2t$ is $N = \{4, 10, 12, 20, 22, 30, 44, 46\}$. If $2t \in N$, then also $P_4(4^a \cdot 2t) = 2$ for integral $a \geqslant 0$, so that $S^{(4)}_{4,2} = \{n \mid n = 4^a n', a \geqslant 0, n' \in N\}$. Written explicitly,

$$S^{(4)}_{4,2} = \{5.2^a, 11.2^a, 4^a, 3.4^a, a \geqslant 1; 30.4^a, 46.4^a, a \geqslant 0\}.$$

The other integers of $S^{(3)}_{4,2}$ are all odd, namely $T = \{9, 13, 17, 19, 21, 29, 31, 35, 39, 47, 71\}$. For each of these 11 integers $j \in T$, we verify that $P_4(4j) > 2$, so that $T = S^{(1)}_{4,2}$ and $S^{(2)}_{4,2} = S^{(4)}_{4,2}$ and, consequently, $S_{4,2} = S^{(1)}_{4,2} \cup S^{(4)}_{4,2}$. This, is, essentially, the result of [8], except for the addition of 19, 30, 46 omitted there.

To compute $|S^{(4)}_{4,2}(x)|$ and, hence $|S_{4,2}(x)|$, we observe that it follows from the explicit presentation of $S^{(4)}_{4,2}$ that, for $x \geqslant 46$,

$$\begin{aligned} S^{(2)}_{4,2} &= \left[\frac{\log(x/5)}{\log 2}\right] + \left[\frac{\log(x/11)}{\log 2}\right] + \left[\frac{\log x}{\log 4}\right] + \left[\frac{\log(x/3)}{\log 4}\right] \\ &\quad + \left[\frac{\log(x/30)}{\log 4}\right] + \left[\frac{\log(x/46)}{\log 4}\right] + 2 \\ &= \frac{4}{\log 2}\log x - R(x); \end{aligned}$$

here the $+2$ originates from the two cases with $a = 0$ and

$$R(x) = \frac{\log 5.11}{\log 2} + \frac{\log 3.30.46}{\log 4} - 2 + \frac{\log(x/5)}{\log 2} - \left[\frac{\log(x/5)}{\log 2}\right] + \ldots$$

$$+ \frac{\log(x/46)}{\log 4} - \left[\frac{\log(x/46)}{\log 4}\right]$$

$$= \frac{\log 55}{\log 2} + \frac{\log 4140}{\log 4} + 1 + R_1(x) = 12.7890672\ldots + R_1(x),$$

where

$$R_1(x) = \frac{\log(x/5)}{\log 2} - \left[\frac{\log(x/5)}{\log 2}\right] + \frac{\log(x/11)}{\log 2} - \left[\frac{\log(x/11)}{\log 2}\right] + \frac{\log x}{\log 4}$$

$$- \left[\frac{\log x}{\log 4}\right] + \frac{\log(x/3)}{\log 4} - \left[\frac{\log(x/3)}{\log 4}\right] + \frac{\log(x/30)}{\log 4} - \left[\frac{\log(x/30)}{\log 4}\right]$$

$$+ \frac{\log(x/46)}{\log 4} - \left[\frac{\log(x/46)}{\log 4}\right] - 3$$

It is trivial to observe that $|R_1(x)| < 3$, but one can do better. It is clear that $R_1(4x) = R_1(x)$, so that it is sufficient to compute $R_1(x)$ only for $1 \leqslant x < 4$, say. On this interval $R_1(x)$ is piecewise continuous and increases between its (easily determined) points of discontinuity. The absolute extrema are attained at 3^+ and at 2.5^- (equivalently, 10^-), respectively, so that, in fact, $-1.4493 < R_1(x) < 1.4987$, or simpler (and somewhat less precisely), $|R_1(x)| < 3/2$. By taking into account also that $|S_{4,2}^{(1)}(x)| = 11$ for $x \geqslant 71$, we conclude that

$$\text{(10)} \qquad |S_{4,2}(x)| = \frac{4}{\log 2}\log x - 1.78906\ldots - R_1(x)$$

holds for $x \geqslant 71$.

As an illustration, we may compute directly that $|S_{4,2}(100)| = 25$, observe that (10) yields $|S_{4,2}(100)| = 24.78636\ldots - R_1(100)$, so that $R_1(100) = -.21363\ldots$

For $x = 20$, (10) is no longer valid. Indeed, by direct counting, $|S_{4,2}(20)| = 9$, and (10) reads $|S_{4,2}(20)| = \frac{4}{\log 2}\log 20 - 1.78906.. - R_1(20)$, so that $R_1(20) = 6.49864.. > 3/2$.

If we set $R_0 = \max\limits_{1 \leqslant n \leqslant 71} \left|\frac{4}{\log 2}\log x - 1.78906 - S_{4,2}(x)\right| = 9.21094\ldots$

(value attained for $n = 8$ only), then (10) holds for all $x \geqslant 1$, but with $|R_1(x)| \leqslant R_0$, rather than $|R_1(x)| < 3/2$.

(c) For $k = 3$, let us first find $S_{3,2}(100)$. For that, we eliminate from the integers $0 < n \leqslant 100$, first the 12 integers with $n \equiv 7 \pmod 8$, for which $P_3(n) = 0$; next, the 24 integers $n \leqslant 100$ with $P_3(n) = 1$ (this set is known; see [2], or—after a few completions—[8]). From the remaining 64 integers, we eliminate the 25 integers divisible by 4 and remain with 39 integers. Of these, the 12 integers of the set $\{41, 50, 54, 65, 66, 74, 81, 86, 89, 90, 98, 99\}$ have $P_3(n) > 2$, so that we remain with the set

$$T = \{9, 17, 18, 25, 26, 27, 29, 33, 34, 38, 45, 49, 51, 53,$$
$$57, 59, 61, 62, 69, 73, 75, 77, 82, 83, 85, 94, 97\},$$

of 27 integers. For all $n \in T$, $P_3(n) = P_3(4^a n) = 2$ and all $n \not\equiv 0 \pmod 4$, $n \leqslant 100$ with $P_3(n) = 2$ are in T. By adding to T the set $T_1 = \{36, 68, 72, 100\}$ of integers $0 < n = 4^a t \leqslant 100$, $t \in T$, we, obtain $S_{3,2}(100) = T \cup T_1$, with $|S_{3,2}(100)| = 31$. From this computation, such results as $|S_{3,2}(37)| = 10$, etc., are obvious.

In order to obtain $S_{3,2}$, we have to determine the finite set M_2 of integers $n \not\equiv 0 \pmod 4$, with $P_3(n) = 2$. For this, we have to determine n_0 and n'_0 such that $h(-n) > 4$ for $n > n_0$, $n \equiv 3 \pmod 8$ and $h(-4n) > 8$, for $n > n'_0$, $n \equiv 1, 2, 5, 6 \pmod 8$. If we assume that Buell's list [3] is complete, then $n_0 = 1555$ and $n'_0 = 862$; if [3] is not complete, any n with $h(-n) \leqslant 4$, $n > 1555$, $n \equiv 3 \pmod 8$, or $h(-4n) \leqslant 8$, $n > 862$, $n \equiv 1, 2, 5, 6 \pmod 8$, respectively, exceeds 10^6. The existence of such an integer, while not disproven, is highly unlikely. Assuming the completeness of [3], we have to select among the integers $n \equiv 3 \pmod 8$, $n \leqslant 1555$, the set V_1 of those integers, with $P_3(n) = 2$; similarly, among the integers $n \equiv 1, 2, 5, 6 \pmod 8$, $n \leqslant 862$, the set V_2 of integers with $P_3(n) = 2$. Then $M_2 = V_1 \cup V_2$ and $S_{3,2} = \{n \mid n = 4^a t, t \in M_2\}$. The numerical work involved in this (still only tentative) determination of M_2 does not appear warranted.

(d) For $k = 2$, let us determine $S_{2,2}$ and $|S_{2,2}(x)|$ for large x. Here $m = 2$, so that (1) becomes

$$\prod(b_i + 1) = 4, \quad \text{or} \quad \prod(b_i + 1) = 3\,.$$

There are three solution sets, namely (i) $b_1 = 3$; (ii) $b_1 = b_2 = 1$; and (iii) $b_1 = 2$. It follows that

$$S_{2,2}^{(3)}(x) = \{n \mid n = 2^a n_2^2 p^3\}, \; S_{2,2}^{(1,1)} = \{n \mid n = 2^a n_2^2 p_1 p_2\},$$

and $S^{(2)}_{2,2} = \{n \mid n = 2^a n_2^2 p^2\}$, with $S_{2,2} = S^{(3)}_{2,2} \cup S^{(1,1)}_{2,2} \cup S^{(2)}_{2,2}$. Next, by Theorem 2(d), $| S^{(3)}_{2,2}(x) | \doteq C_1 \left(\frac{x}{\log x}\right)^{1/2}$, $| S^{(1,1)}_{2,2}(x) | \doteq C_3 \frac{x}{\log x} \log_2 x$, with $C_3 = \frac{1}{2} \prod_{q \equiv 3} (1-q^{-2})^{-1} \simeq .5840...$ and $| S^{(2)}_{2,2}(x) | \doteq C_2 \left(\frac{x}{\log x}\right)^{1/2} \log_2 x$.

We conclude that

$$S_{2,2} = \{n \mid n = 2^a n_2^2 n_1, \quad \text{with } n_1 = p^3, p_1 p_2, \quad \text{or } p^2\}.$$

The dominant term is

$$C_3 \frac{x}{\log x} \log_2 x \simeq .5840.. \frac{x}{\log x} \log_2 x = g(x) \tag{11}$$

say. The proof suggests that, even if the weak asymptotic equality is a genuine asymptotic equality, one should expect the ratio of the two sides of (11) to approach unity only for rather large values of x. The following tabulation seems to bear this out.

x	$\mid S_{2,2}(x) \mid$	$g(x)$	ratio $g(x)/\mid S_{2,2}(x) \mid$
10	0	2.115	∞
10^2	5	19.365	3.873
10^3	74	163.390	2.208
10^4	822	1,407.840	1.713
10^5	8454	12,394.650	1.466
10^6	82022	110,995.715	1.353
10^7	781073	1,007,244.600	1.28956

The function represented by the last ratio appears to be fairly well interpolated by the curve $1 + 1.6/\log x + 43/\log^2 x$. This indicates that, in order to reduce said ratio even only to 1.1, it would be necessary to go beyond $x = 3.10^{13}$ and further numerical experimentation is not warranted.

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