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LARGE FREE GROUPS OF ISOMETRIES AND THEIR GEOMETRICAL USES

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§ 1. INTRODUCTION

Various geometric constructions related to the Banach-Tarski paradoxical decomposition of the sphere require the existence of free groups of isometries acting without fixed points (i.e., no nonidentity element of the group has a fixed point) or in a locally commutative way (i.e., if two group elements have a common fixed point, then they commute). For some of these constructions a free group of rank 2 is sufficient; others require one of rank $2^{\aleph_0}$. It is the purpose of this paper to fill a few gaps in this subject, where the underlying spaces are the spheres S^n, the Euclidean spaces $\mathbf{R}^n$, and the hyperbolic spaces H^n.

For groups of rank 2, all cases of this problem have been solved, and we shall review these results in § 2. For the case of rank $2^{\aleph_0}$, we present a unified approach (§ 4-6) to the known results which is sufficiently general to settle the heretofore unresolved cases, H^3, S^{4n+1} and S^4. The main idea of our proofs is a general topological technique (introduced in [25]) that uses the groups of rank 2 to obtain a perfect (i.e., closed and without isolated points) set of free generators. In all cases except H^2, the existence of a fixed-point free or locally commutative rank 2 free group of isometries implies the existence of a group of rank $2^{\aleph_0}$ with the same properties.

In § 7 and § 8 we discuss the geometric consequences of the existence of large free groups of isometries. For example, each of $S^n\,(n \geqslant 2)$, $H^n\,(n \geqslant 2)$ and $\mathbf{R}^n\,(n \geqslant 3)$ contains a set which is, simultaneously, a third, a quarter, ..., a $2^{\aleph_0}$ th part of the space. In § 8 we show how paradoxical decompositions of $H^n\,(n \geqslant 2)$ can be constructed using Borel sets (and not using the Axiom of Choice). However, such paradoxical decompositions of $\mathbf{R}^n$ either do not exist, even allowing arbitrary sets ($n = 1$ or 2), or exist ($n \geqslant 3$), but require nonmeasurable sets and the Axiom of Choice.

Finally, in § 9 we discuss what can be done in $\mathbf{R}^2$ if we allow area-preserving linear or affine transformations instead of just isometries.

Proofs of several of the results mentioned or used in this paper, such as Theorems 3, 4 (b) and (c), 5, 6, and 7, may be found in [41].

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§ 2. The Main Theorem

The action of a group, G, on a set, X, is called *fixed-point free* if $g(x) \neq x$ for all $x \in X$ and $g \in G \setminus \{I\}$ (I is the identity of G). The action is called *locally commutative* if, for each $x \in X$, $\{\sigma \in G : \sigma(x) = x\}$ is a commutative subgroup of G; equivalently, if two elements of G have a common fixed point in X then they commute. For any group G and any abstract (reduced) group word w in m variables, the function $f_w : G^m \to G$ is defined by $f_w(\sigma_1, ..., \sigma_m) = w(\sigma_1, ..., \sigma_m)$.

If X is a metric space and also an oriented manifold, then $G(X)$ denotes the group of orientation-preserving isometries of X, with its natural topology. In particular, $G(S^n) = SO_{n+1}$, $G(H^2) = PSL_2(\mathbf{R})$ and $G(H^3) = PSL_2(\mathbf{C})$.

A set in a complete metric space is called *perfect* if it is nonempty, closed and without isolated points; a perfect set has at least $2^{\aleph_0}$ elements.

THEOREM 1.

(a) *Each of the groups* $G(S^n)$, *where* n *is odd and* $n \geqslant 2$, $G(\mathbf{R}^n)$, *where* $n \geqslant 3$, *and* $G(H^n)$, *where* $n \geqslant 3$, *has a free subgroup with a perfect set of free generators whose action on the space is fixed-point free.*

(b) $G(H^2)$ *has a discrete free subgroup of rank* 2 *(and hence also rank* $\aleph_0$*) which is fixed-point free, but no such free subgroup of* $G(H^2)$ *can have uncountable rank.*

(c) $G(H^2)$ *and each of the groups* $G(S^n)$, $n \geqslant 2$, *have locally commutative free subgroups with a perfect set of free generators.*

The above theorem is false in all omitted dimensions. This is because the isometry groups in the low dimensions are all solvable, and hence contain no free subgroup of rank 2. Also, each element of SO_{2n+1} has a fixed point on S^{2n}, and this is why part (a) fails for spheres of even